FRACTAL SYSTEM IDENTIFICATION FOR ROBUST CONTROL – THE MOMENT APPROACH

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Abstract: In the last decade, several authors focus on fractional-order systems (called also fractal systems), which can describe reality better than classical integer-order models. The fractional-order description is based on generalized fractional order derivative and integral operators. This paper deals with fractional order system identification for the purpose of a robust controller design. The proposed method combines two pieces of information: *a priori* information and experimental data. Firstly, the class of all admissible process models is defined as the set of all *a priori* admissible transfer functions which are consistent with experimental data (moments of the system). Using a value set approach, the parameterization of all so called extreme transfer functions (those that are important for robust design) is presented.

Keywords: fractional order systems, identification, moments, value set, robust design

1. Introduction

The fractional calculus (*FC*) based on generalized fractional order integro-differential operators is a very old topic in mathematics. There are many physical phenomena whose behavior can be described using fractional calculus. Let us mention transmission lines, electrochemical processes, dielectric polarization, heating processes and viscoelastic materials. With growing power of computers *FC* becomes a more actual topic also in control theory [2]. That is why several practical engineering applications of *FC* were developed in the last decades [4, 5].

This paper deals with fractal system identification based on the method of moments [6]. Advantages of this approach are described in [1]. The paper is organized as follows: In Section 2, the main characteristics of fractal systems are reminded. In Section 3, the idea of moment system description and process model set [1] as an outcome of identification procedure is given. Extreme transfer functions parameterization is presented in Section 4. Finally, Section 5 contains concluding remarks and some ideas for further work.

2. Fractal systems

Fractal systems are known to exhibit a fractional power function dependence on frequency, or equivalently a fractional slope on the log-log plot. However, in most cases, the system usually exhibits finite magnitude at very low frequencies. Hence, a fractional power pole is a more appropriate representation of its frequency behavior. In general, a typical fractal system consists of many fractional slopes in the entire frequency range. Such a system can be described by the transfer function in the form

$$F(s) = \frac{K}{\prod_{i=1}^{p} (t_i s + 1)^{n_i}}, \qquad t_i > 0, i = 1, \mathbf{K}, p, \tag{1}$$

where K >0, p is an arbitrary integer and $t_i, n_i, \quad i = 1, 2, \mathbf{K}, p$ are real numbers. Note, that if $p \to \infty$ then all integer order systems in the form

$$F(s) = \frac{Ke^{-Ds}}{\prod_{i=1}^{\infty} (t_i s + 1)} \tag{2}$$

are particular cases of the transfer function (1).

3. Moment system description and process model set

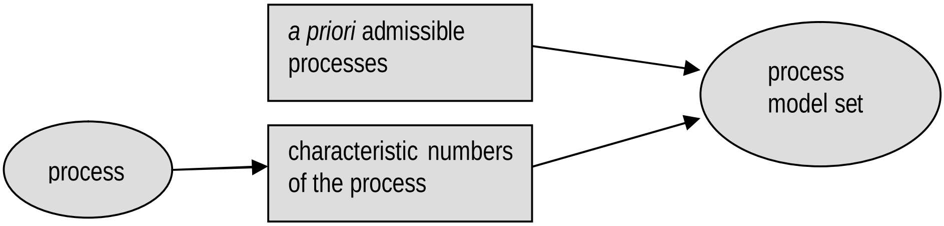


Fig. 2. The main idea of process model set

A priori admissible processes

The most of industrial processes are monotone (have a monotone step response). Pressure, temperature or concentration processes belong into this category. These processes are usually modeled by low integer order models with dead time although the reality is often fractional. Therefore, it is reasonable to consider process models in the form (1).

Characteristic numbers of the process

There are many possibilities for process description. Our description is based on first three moments of the impulse response

$$m_i = \int_0^\infty t^i h(t) dt, \qquad i = 0, 1, 2, \tag{3}$$

where $h(t)$ is the impulse response of the process at hand. The numbers m_0, m_1, m_2 can be substituted by another numbers κ, μ, σ^2 defined as

$$k = m_0, \qquad m = \frac{m_1}{m_0}, \qquad s^2 = \frac{m_2}{m_0} - \frac{m_1^2}{m_0^2} \tag{4}$$

How to measure the moments (3) on real processes is described in [1]. Note, that for the fractal system in the form (1) following relations can be obtained

$$k = K, \qquad m = \sum_{i=1}^{p} t_i n_i, \qquad s^2 = \sum_{i=1}^{p} t_i^2 n_i. \tag{5}$$

Definition 1 (Process Model Set). *The transfer function F(s) is admissible, if following conditions are satisfied*

(i) Transfer function F(s) is in the form (1), $n_i \ge m$, $i = 1, 2, \mathbf{K}, p$, $\sum_{i=1}^{p} n_i \le n$

(ii) Transfer function F(s) satisfies conditions (5)

The set of all admissible systems will be called process model set and will be denoted by $S_p^{n,m}(k, m, s^2)$.

Lemma 1. *Let* $n \ge 2m$, *then the model set* $S_p^{n,m}(K_0, m, s^2)$ *is not empty iff*

$$\frac{1}{n} \le \frac{s^2}{m^2} \le \frac{1}{m}. \tag{6}$$

Moreover, there exist infinitely many members in the model set, if both inequalities in (6) are strict. The proof is omitted for brevity.

4. Extreme transfer function parameterization

Definition 2 (Value set). *The set* $V_p^{n,m}(w) = \{F(jw) : F(s) \in S_p^{n,m}(k, m, s^2)\}$ *in the complex plane will be called the value set of the model set* $S_p^{n,m}(k, m, s^2)$ *at frequency* $\omega > 0$.

Definition 3 (Extreme Transfer Function). *The transfer function* $F(s) \in S_p^{n,m}(k, m, s^2)$ *is called extreme if there exists* $\omega > 0$, *such that* $F(jw) \in \partial V_p^{n,m}(w)$, *where* ∂V *denotes the boundary of the value set.*

Without a loss of generality we can normalize processes in gain and in time and consider the normalized process model set $S_p^{n,m}(1, 1, s^2)$. The following theorem can be proved.

Theorem 1 (Extreme transfer function parameterization) *All extreme transfer functions of the model set* $S_p^{n,m}(1,1,s^2)$ *for any* $p \geq 2$ *can be expressed either in the form*

$$F(s)=\frac{1}{(t_1(a)s+1)^{n_1(a)}(t_2(a)s+1)^{n_2(a)}}\quad , \tag{7}$$

where

$$t_1(a)=\frac{1-\sqrt{\frac{n_2}{n_1}\left(s^2(n_1+n_2)-1\right)}}{(n_1+n_2)},\quad t_2(a)=\frac{1+\sqrt{\frac{n_1}{n_2}\left(s^2(n_1+n_2)-1\right)}}{(n_1+n_2)},$$

and n_1, n_2 *range according to one of the following rules*

(i) $n_1(a)=m,\ n_2(a)=a,\quad a\in\left[\max\left\{m,\frac{1-ms^2}{s^2}\right\},\min\left\{n-m,\frac{1}{s^2}\right\}\right]$

(ii) $n_1(a)=a,\ n_2(a)=m,\quad a\in\left[\max\left\{m,\frac{1-ms^2}{s^2}\right\},n-m\right]$

(iii) $n_1(a)=n-a,\ n_2(a)=a,\quad a\in\left[m,\min\left\{n-m,\frac{1}{s^2}\right\}\right],$

or in the form

$$F(s)=\frac{1}{(t_1(a)s+1)^m(t_2(a)s+1)^m(t_3(a)s+1)^{n-2m}}\quad , \tag{8}$$

where

$$t_1(a)=\frac{1-a(n-2m)-\sqrt{(n-2m)a(2-na)+2ms^2-1}}{2m}$$

$$t_2(a)=\frac{1-a(n-2m)+\sqrt{(n-2m)a(2-na)+2ms^2-1}}{2m}$$

$$t_3(a)=a\ ,$$

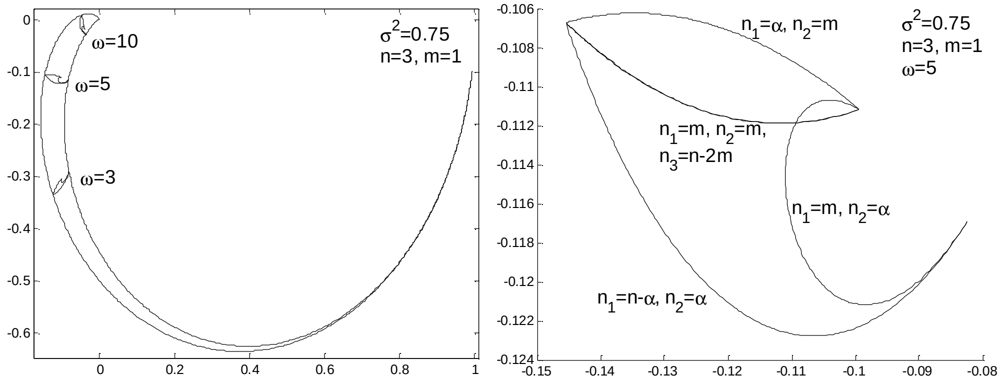
$$a\in I_1\cup I_2,\quad I_1=\left[\max\{0,a_1,a_2\},\min\{b_1,b_3\}\right],\ I_2=\left[\max\{0,a_1\},\min\{b_1,b_2,b_3\}\right],$$

$$a_1=\frac{n-2m-\sqrt{2m(n-2m)(ns^2-1)}}{n(n-2m)},\quad a_2=\frac{n-2m+\sqrt{m(n-2m)[(n-m)s^2-1)}}{(n-m)(n-2m)}$$

$$b_1=\frac{n-2m+\sqrt{2m(n-2m)(ns^2-1)}}{n(n-2m)},\quad b_2=\frac{n-2m-\sqrt{n(n-2m)[(n-m)s^2-1]}}{(n-m)(n-2m)}$$

$$b_3=\frac{1}{n-2m}.$$

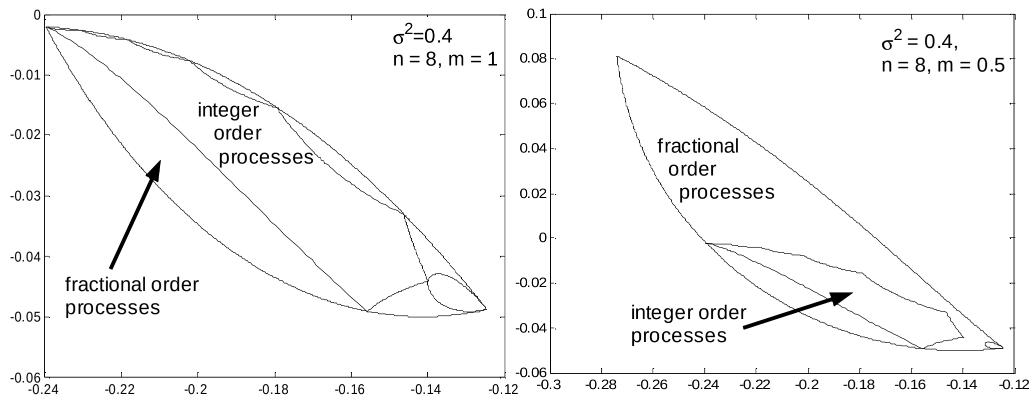
The proof is omitted for brevity. Note, that the extreme transfer functions are not dependent on frequency ω. They are also not dependent on the number of fractional poles *p*. An example of value set boundary is given in Fig. 3.



(a) (b)

Fig. 3. Example of the value set a) general view b) detailed value set for ω=5

In Fig. 4a, you can see how the value set enlarges if fractional order processes instead of integer order processes are considered. This difference will be much bigger if we decrease the minimum order *m* such that it is close to zero (Fig. 4b). Usually, only the processes generating the vertexes of the value set boundary are important for robust controller design. In that case, controller design can be done for finite number of extreme transfer functions and design specifications will be fulfilled for the whole model set. In general case, it is sufficient to consider the processes given in Theorem 1 in robust design procedure.



(a) (b)

Fig. 4. Comparison between integer and fractional order processes

5. Conclusions

This article deals with moment identification of fractional order systems for robust control. The moment approach is suitable for processes with monotone step response. The outcome of identification is explicit parameterization of all extreme transfer functions. Using this parameterization, the corresponding value set may be computed for any $w > 0$ and used in the robust design procedure. The further step in current research is to develop a new design method based on the results of [3] for fractal systems.

This work was particularly supported by MŠMT ČR – project no. MSM 235200004 and GAČR – project no. 102/02/0425.

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