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Analytical Design of a Wide Class of Controllers with Two Tunable Parameters Based on H_∞ Specifications

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Motivation for the development of a new method for the H_∞ design of controllers with a fixed structure

Design a fixed PI controller for a linear stable system with the transfer function in the form

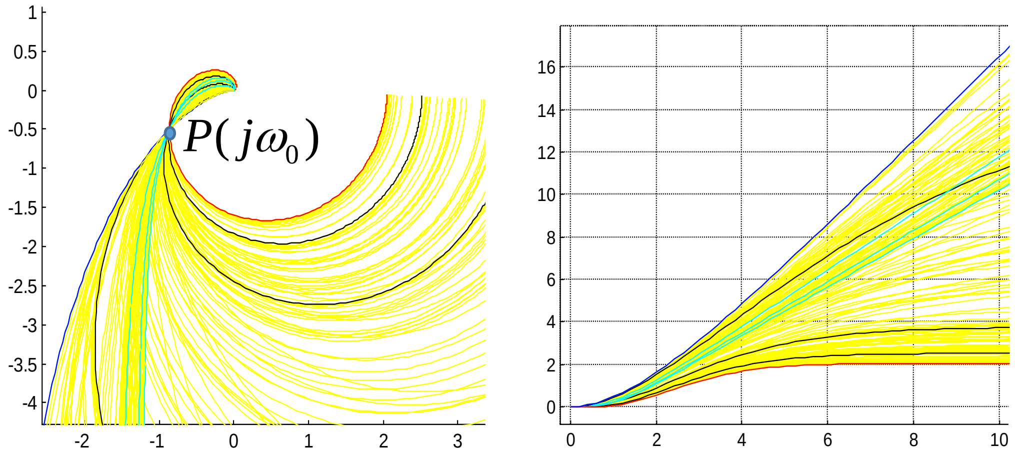
$$P(s)=\frac{k_P}{s^l(\tau_1 s+1)(\tau_2 s+1)\cdot\ldots\cdot(\tau_N s+1)},\quad l\in\{0,1\},\quad \tau_i>0,\quad N \text{ is arbitrary integer,}$$

based on the knowledge of only the single point of the frequency response $P(j\omega)$ at frequency ω_0, that the sensitivity function of the closed loop satisfies the condition

$$\forall\omega:\quad |S(j\omega)|\triangleq\left|\frac{1}{1+C_{PI}(j\omega)P(j\omega)}\right|\le M_S$$

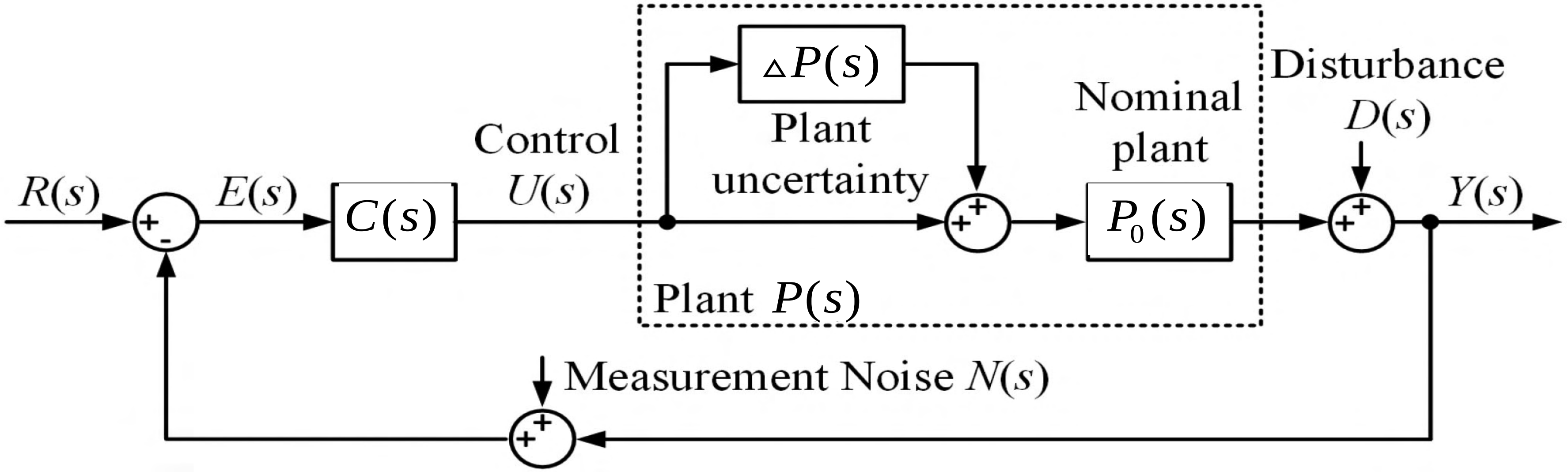
$$\Updownarrow$$

$$\|S(s)\|_\infty\le M_S$$



for all transfer functions $P(s)$ described above.

A Typical Feedback Control Configuration Robustness/Performance Channels



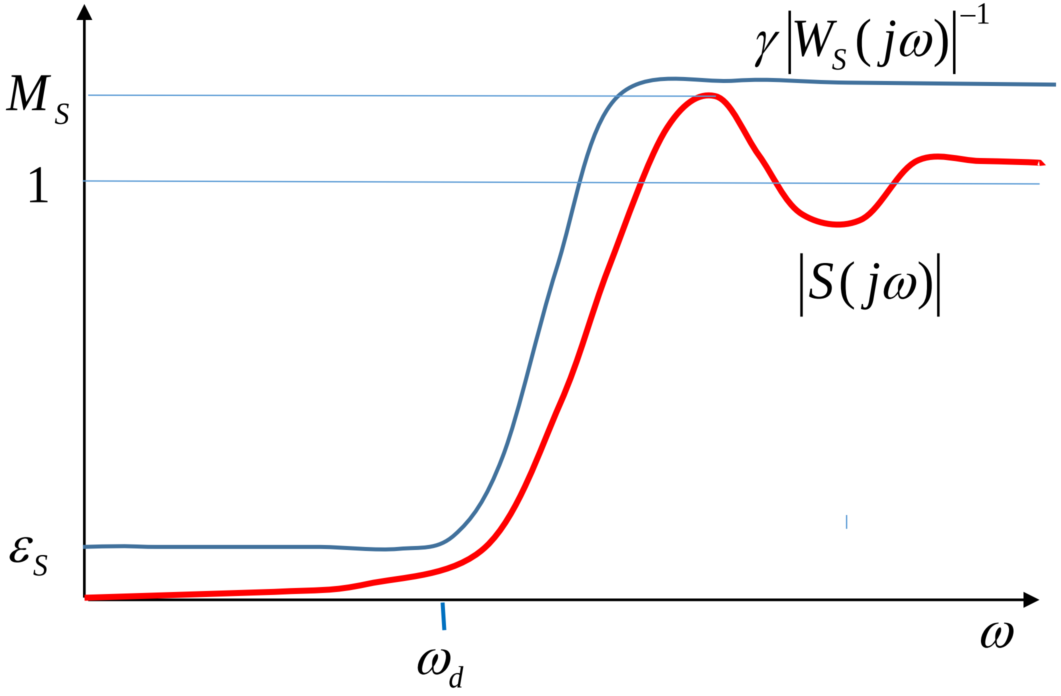
$$Y(s) = T(s)\left(R(s) - N(s)\right) + S(s)D(s)$$

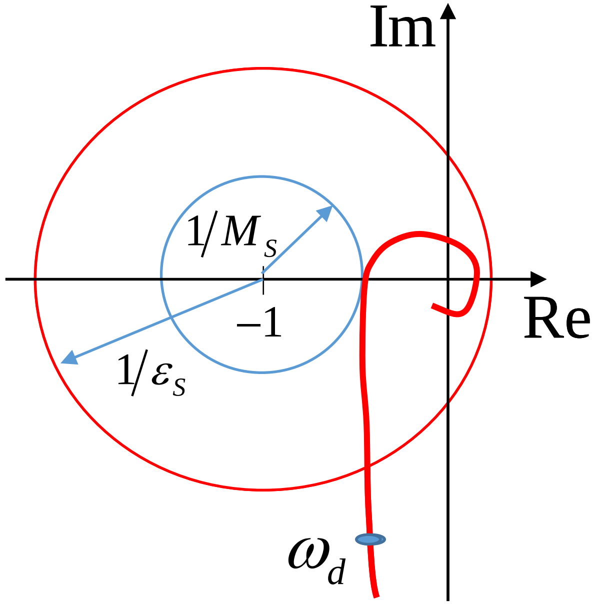
$$E(s) = S(s)\left(R(s) - D(s)\right) + T(s)N(s)$$

The Sensitivity Function	The Complementary Sensitivity Function	The Input Sensitivity Function	The Output Sensitivity Function
$S(s) = \frac{1}{1+C(s)P(s)}$,	$T(s) = \frac{C(s)P(s)}{1+C(s)P(s)}$,	$S_P(s) = \frac{P(s)}{1+C(s)P(s)}$,	$S_C(s) = \frac{C(s)}{1+C(s)P(s)}$

Typical Constraint on Sensitivity Function

$$\|W_S(s)S(s)\|_\infty < \gamma \quad \Leftrightarrow \quad \forall \omega: \ |S(j\omega)| < \gamma |W_S(j\omega)|^{-1}$$





Some History of The H_∞ Control Problem

- The origins of the H_∞ -problem date back to the 1960s, when Zames discovered the small gain theorem and extended classical control techniques to MIMO control architectures.
- In nominal H_∞ synthesis, full-order H_∞ feedback controllers are computed via semidefined programming (1994) or algebraic Riccati equations (1989).
- Analytical H_∞ design of PI controllers (1998).
- Structured H_∞ synthesis in MATLAB (hinfstruct) via nonsmooth optimizers (2011).

The Affine-structured Controller (AF-controller)

$$C_{AF}(s,k) \triangleq k_q Q(s) + k_r R(s) + F(s),$$

where $Q(s), R(s),$ and F(s) are arbitrary proper rational transfer functions; $k_q,$ and k_r are tunable controller parameters, and $k = \left[k_r, k_q\right]^T \in \mathbb{R}^2$ is the vector parameter.

Typical examples :

PI controller: $Q(s)=1,\ R(s)=\frac{1}{s},\ F(s)=0;$ $C_{PI}(s)=k_q+k_r\frac{1}{s}$

PD controller: $Q(s)=1,\ R(s)=\frac{s}{\tau s+1},\ F(s)=0;$ $C_{PD}(s)=k_q+k_r\frac{s}{\tau s+1}$

PR controller: $Q(s)=1,\ R(s)=\frac{2\omega_c s}{s^2+2\omega_c s+\omega_0^2},\ F(s)=0;$ $C_{PR}(s)=k_q+\frac{2\omega_c s k_r}{s^2+2\omega_c s+\omega_0^2}$

PID controller: $Q(s)=\frac{1}{s},\ R(s)=\frac{s}{\tau s+1},\ F(s)=k_p;$ $C_{PID}(s)=k_p+k_q\frac{1}{s}+k_r\frac{s}{\tau s+1}$

Other parameters except the two tunable ones must be fixed!

Other Possible Choices of Q(s), R(s), and F(s)

- Converting the selected controller to an affine controller.

Example: $C_{LL}(s) = k \cdot \dfrac{T_1 s + 1}{T_2 s + 1} \rightarrow C_{AF}(s) = k_q + k_r \dfrac{s}{T_2 s + 1}, \; Q(s) = 1, \; R(s) = \dfrac{s}{T_2 s + 1}$

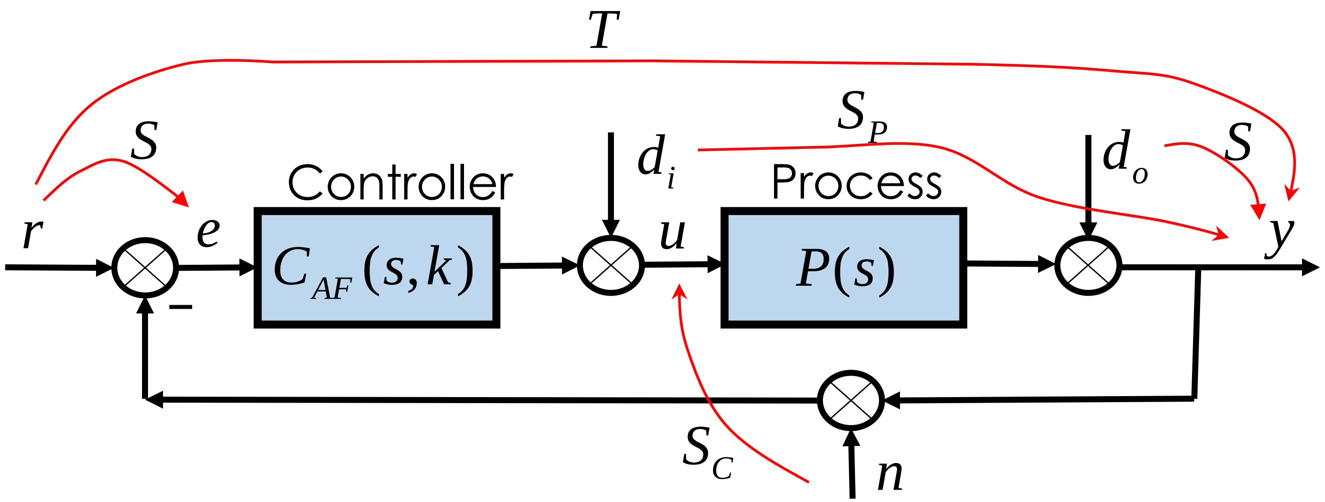
- Selection of $Q(s), R(s),$ and $F(s)$ based on the Internal Model Principle.

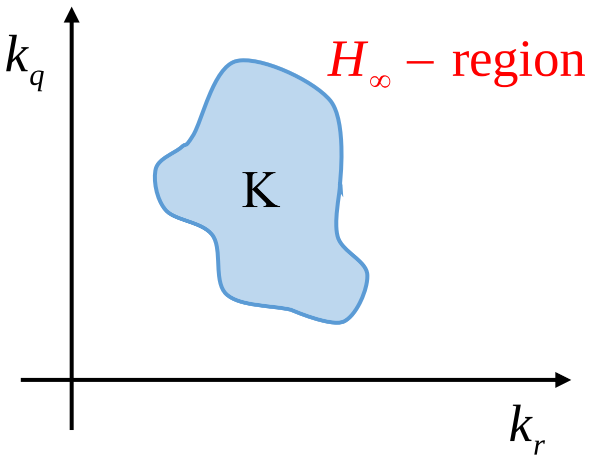
Example: PR-controller (High gain of $R(s) = \dfrac{2\omega_c s}{s^2 + 2\omega_c s + \omega_0^2}$ at the frequency ω_0)

- Increasing the control performance by increasing the order of the existing controller C_0(s).

Example: $C_{new}(s) = C_0(s) + k_q Q(s) + k_r R(s) \rightarrow F(s) = C_0(s)$

Basic H_∞ Formulation of Affine-structured Controller Design Problem: Single H_∞ Constraint





$H(s,k=[k_q,k_r]) \quad W(s)S_*(s,k),\ S_* \in \{S,T,S_C,S_P\}$ .... weighted sensitivity function

$\mathrm{K} \triangleq \left\{ \left[k_q,k_r\right]:\ \|H(s,k)\|_\infty \triangleq \sup_\omega |H(j\omega,k)| \le \gamma,\ \text{the closed-loop is internally stable} \right\}$ ...... H_∞-region in the parameter plane

1) Find the H_∞- region K in the k_r-k_q plane.

2) Find the optimal AF-controller in the H_∞- region K with respect to some criterion, e.g. $IAE \triangleq \int_0^\infty |e(t)|\,dt$ for the step in the reference value r (servo problem) or load disturbance d_i (regulator problem).

D-decomposition for one H_∞ Constraint $\|H(s,k)\|_\infty < \gamma$

$$\mathrm{K} \triangleq \{k : |H(j\omega,k)| < \gamma,\ \forall \omega \in [0,\infty)\} = \{k : |H_n(j\omega,k)| < \gamma |H_d(j\omega,k)|,\ \forall \omega \in [0,\infty)\}$$

Explicit solution for the case of AF-controller

Theorem 1. [*] The boundary of the set K, called the H_∞ region, is contained in the solution of the systems

$\begin{cases} H_n(j\omega,k)=0, & (1a) \\ H_d(j\omega,k)=0, & (1b) \end{cases}$	(1a), (1b)	Straight lines or empty set in the parametric plane $k_q - k_r$.
$\begin{cases} \lvert H(j\omega,k)\rvert^2 = \gamma^2, & (2a) \\ \dfrac{d\lvert H(j\omega,k)\rvert^2}{d\omega} = 0, & (2b) \end{cases}$	(2a), (2b)	Critical curves $\varphi_k(\omega)$, $k=1,\dots,l,\ l\in\{0,2,4\}$, in the parametric plane $k_q - k_r$.

for $\omega \in [0,+\infty)$ and three equations

$\lvert H(0,k)\rvert = \gamma,$	(3)	Straight lines or empty set in the parametric plane $k_q - k_r$.
$\lvert H(\infty,k)\rvert = \gamma,$	(4)	
$h_d(k) = 0,$	(5)	

where h_d is the coefficient at the higher order of a polynomial $H_d(s,k)$.

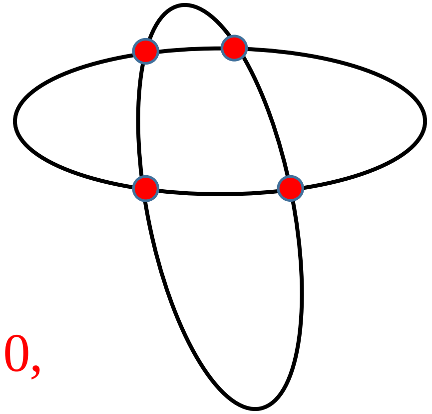
[*] E. N. Gryazina, B. T. Polyak, A. A. Tremba, D-decomposition technique state-of-the-art, Automation and Remote Control, 2008.

Explicit Solution of the system (2a), (2b) for the constraint $\|H(s,k)\|_\infty < \gamma,\ H(s,k) \triangleq S(s,k)$

Equation (2a) $|H(j\omega,k)|^2 = \gamma^2$ is equivalent to $p_1(\omega,k) = 0,$

Equation (2b) $d|H(j\omega,k)|^2 / d\omega = 0$ is equivalent to $p_2(\omega,k) = 0,$

where $p_1(\omega,k)$ and $p_2(\omega,k)$ are second-order polynomials with real coefficients in the variables k_q and k_r.



Proposition. The system of polynomial equations

$$p_1(\omega,k) = a_1(\omega)k_q^2 + b_1(\omega)k_r^2 + c_1(\omega)k_q k_r + d_1(\omega)k_q + e_1(\omega)k_r + f_1(\omega) = 0,$$

$$p_2(\omega,k) = a_2(\omega)k_q^2 + b_2(\omega)k_r^2 + c_2(\omega)k_q k_r + d_2(\omega)k_q + e_2(\omega)k_r + f_2(\omega) = 0,$$

has analytical solutions in radicals. (The solution of this system can be converted to the solution of a polynomial quartic equation with one unknown.) There are two, four, or no real solutions to this system of equations. These solutions determine the parametric curves (critical curves φ_k from Theorem 1) with the parameter ω in the parametric plane of the affine controller.

Main Result: Isolation of H_∞ - Region

The systems of equations (1), (2) and equations (3),(4) and (5) of Theorem 1 have analytical solutions for affine controller $C_{AF}(s)$ and the rational transfer function $P(s)$. These solutions define the critical curves and lines in the parametric plane that divide it into regions.

Example: $H_\infty -$ region for unstable process:

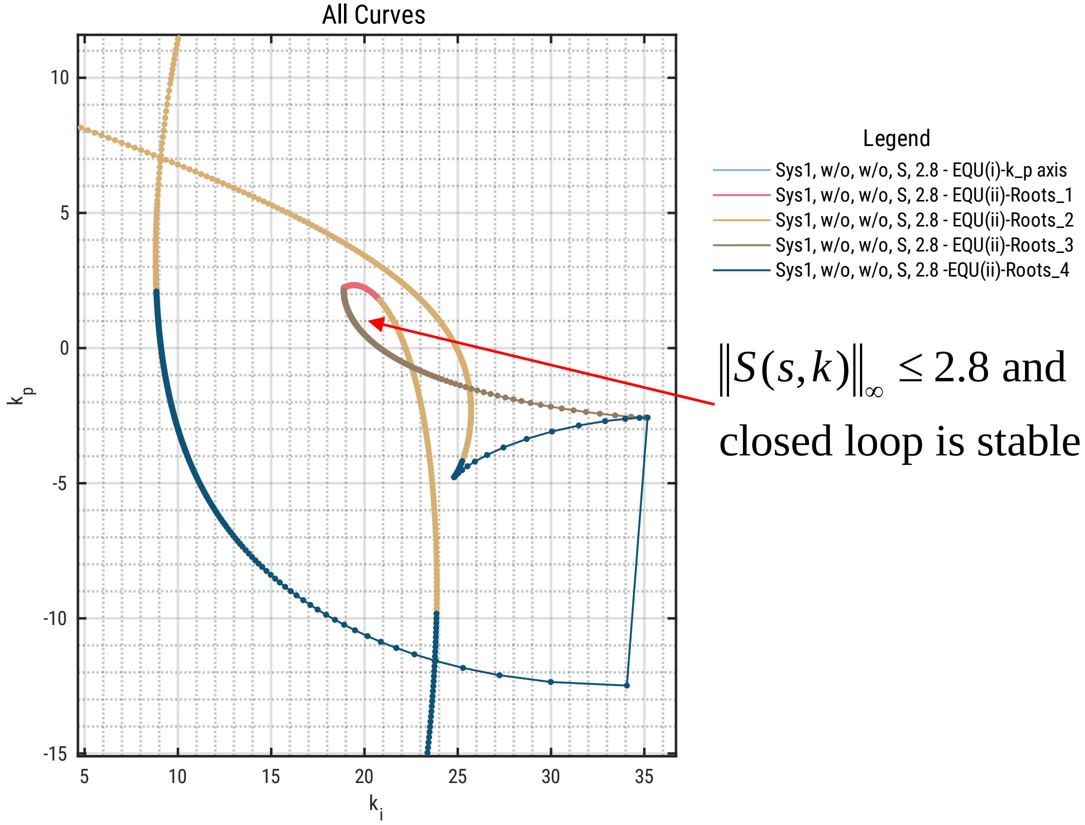
$$P(s) = \frac{s^3 + 4s^2 - s + 1}{s^5 + 2s^4 + 32s^3 + 14s^2 - 4s + 50},$$

$$C_{AF}(s,k) = C_{PI}(s,k) = k_p + \frac{k_i}{s},$$

$$k_p = k_q,\ k_i = k_r,$$

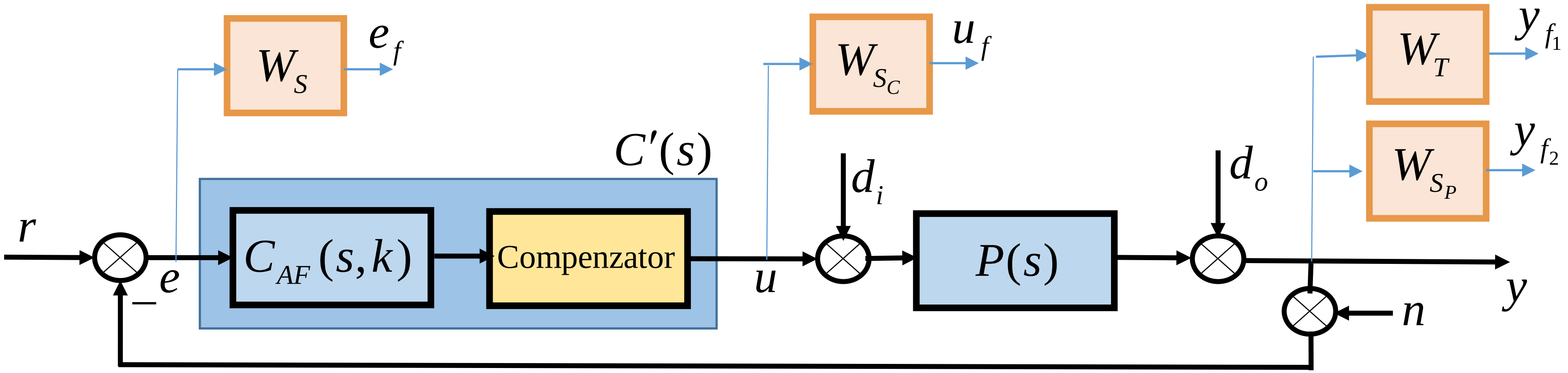
$$Q(s) = 1,\ R(s) = \frac{1}{s},\ F(s) = 0.$$

$$\left\| S(s,k) \right\|_\infty = \left\| \frac{1}{1 + C(s,k)P(s)} \right\|_\infty \le M_S = 2.8$$



$\left\| S(s,k) \right\|_\infty \le 2.8$ and closed loop is stable

H_∞ Multiple Constraints



The performance or robustness channel

$$H_i \equiv H_i(P,C') = W_i S_{*i}, \quad i = 1,..,k$$

is a scalar weighted closed-loop sensitivity function.

Find all controllers C_{AF} for which it holds

$$\left\| H_i(P,C') \right\|_\infty \le \gamma_i, \; i = 1,..,k$$

subject to C' stabilizes P internally

$$C' \triangleq C_{AF}(s,k) \cdot C_{comp}.$$

Multi-model Design Objectives

H_∞ constraints on the robust/performance channels

The multi-model set:

$$\mathrm{P} \triangleq \{P_1, P_2, \ldots, P_n\}$$

Design objectives:

$$\text{(i)} \quad \forall P \in \mathrm{P}: \ T(s) \text{ is stable} \ \wedge \ \|W_T(s)T(s)\|_\infty < \gamma_T,$$

$$\text{(ii)} \quad \forall P \in \mathrm{P}: \ S(s) \text{ is stable} \ \wedge \ \|W_S(s)S(s)\|_\infty < \gamma_S,$$

$$\text{(iii)} \quad \forall P \in \mathrm{P}: \ S_P(s) \text{ is stable} \ \wedge \ \|W_{S_P}(s)S_P(s)\|_\infty < \gamma_{S_P},$$

$$\text{(iv)} \quad \forall P \in \mathrm{P}: \ S_C(s) \text{ is stable} \ \wedge \ \|W_{S_C}(s)S_C(s)\|_\infty < \gamma_{S_C},$$

where W_S, W_T, W_{S_P}, and W_{S_C} are stable weighting functions,

$\|H\|_\infty \triangleq \sup_\omega |H(j\omega)|$ is called H_∞- norm.

Solving the Motivational Task

Input:

Model Set:

$\omega_0 = 1,$

$P(j\omega_0) = e^{j*1.8},$

$N = 10,$

$\mathrm{P} \triangleq \{P_1, P_2, \ldots, P_{66}\},$

$P_i, i = 1,..,66$ samples

of extremal systems*

Design specification:

PI-controller: $C_{PI}(s) = k_p + k_i \cdot 1/s$

$\forall P \in \mathrm{P}: \ \|S(s,k)\| \leq 1.4,$

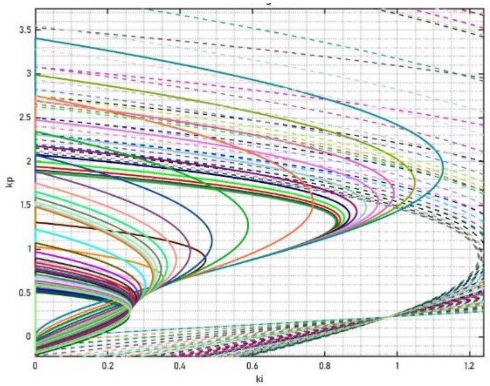
maximum integral gain

Output:

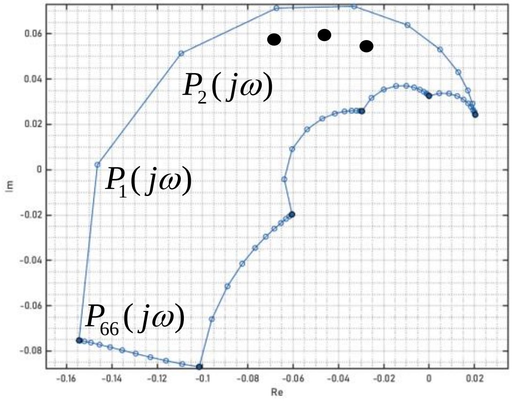
$k_p = 0.4725$

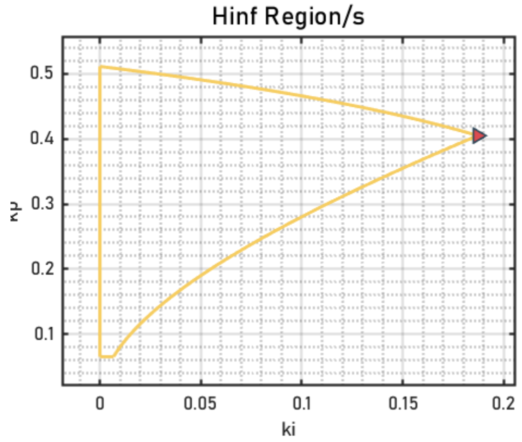
$k_i = 0.3109$

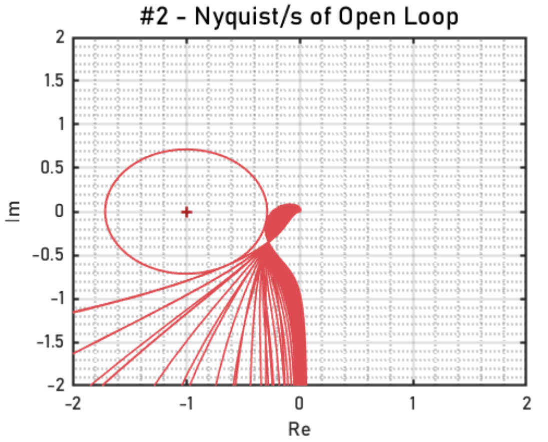
The intersection of all H_∞ regions

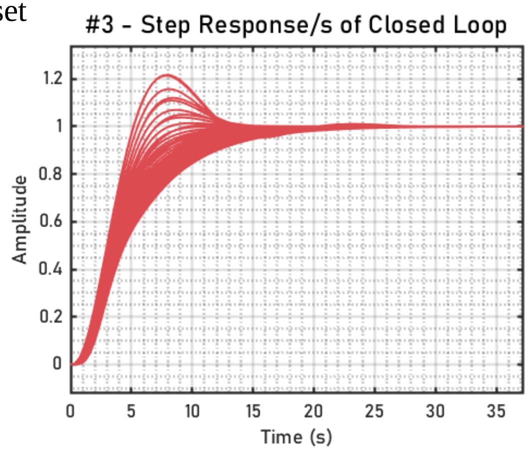


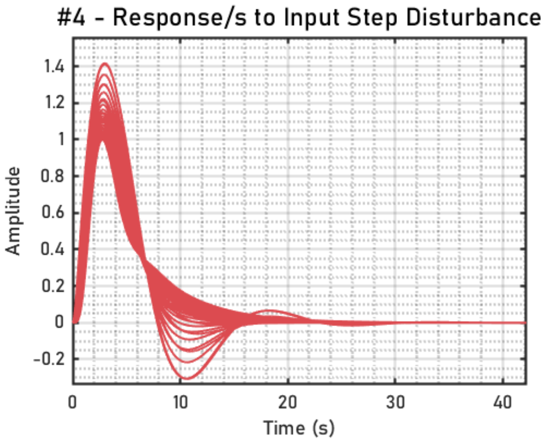
The boundary samples of the process value set









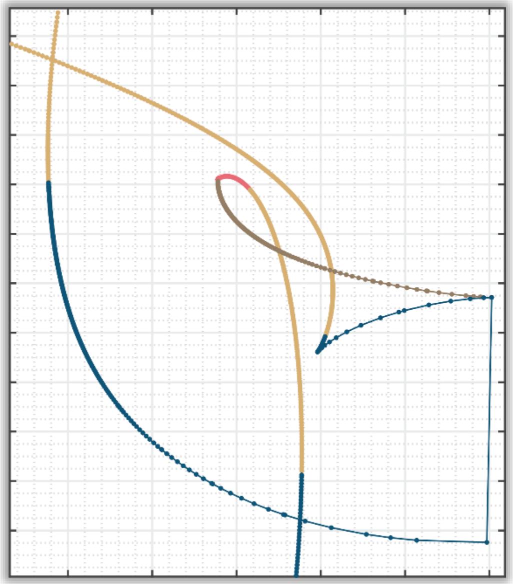


Sketch of H_∞ region isolation algorithm

Single Specification Problem

STEP 1

Finding all critical points/curves in the parametric plane

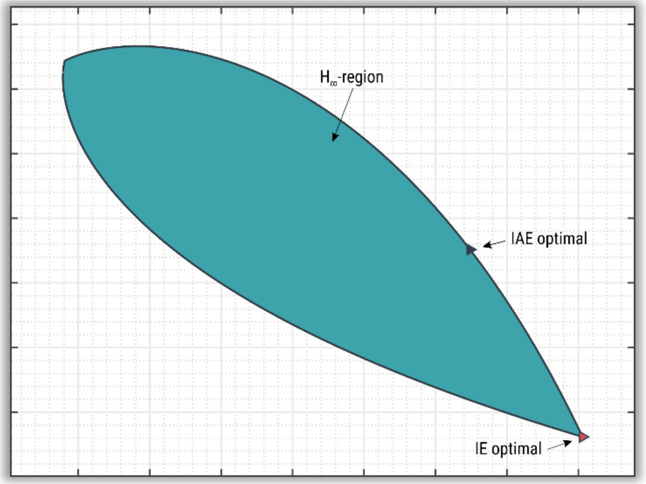


STEP 2

Select parts that meet Hinf specifications and the condition of internal stability

STEP 3

Create a border of region in a parametric plane

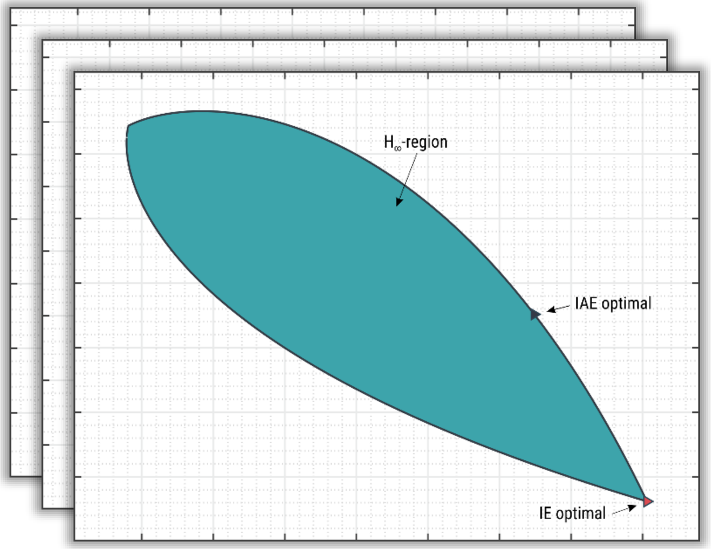


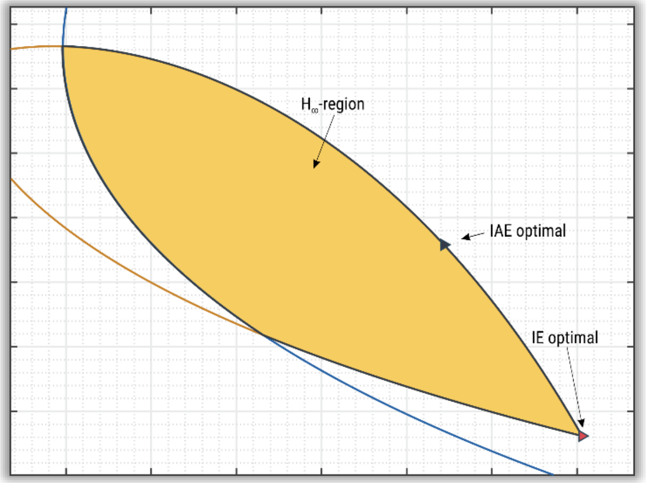
Sketch of H_∞ region isolation algorithm

Multiple Specification Problem

STEP 3i

Create a border of regions in a parametric plane for all specifications





STEP 4

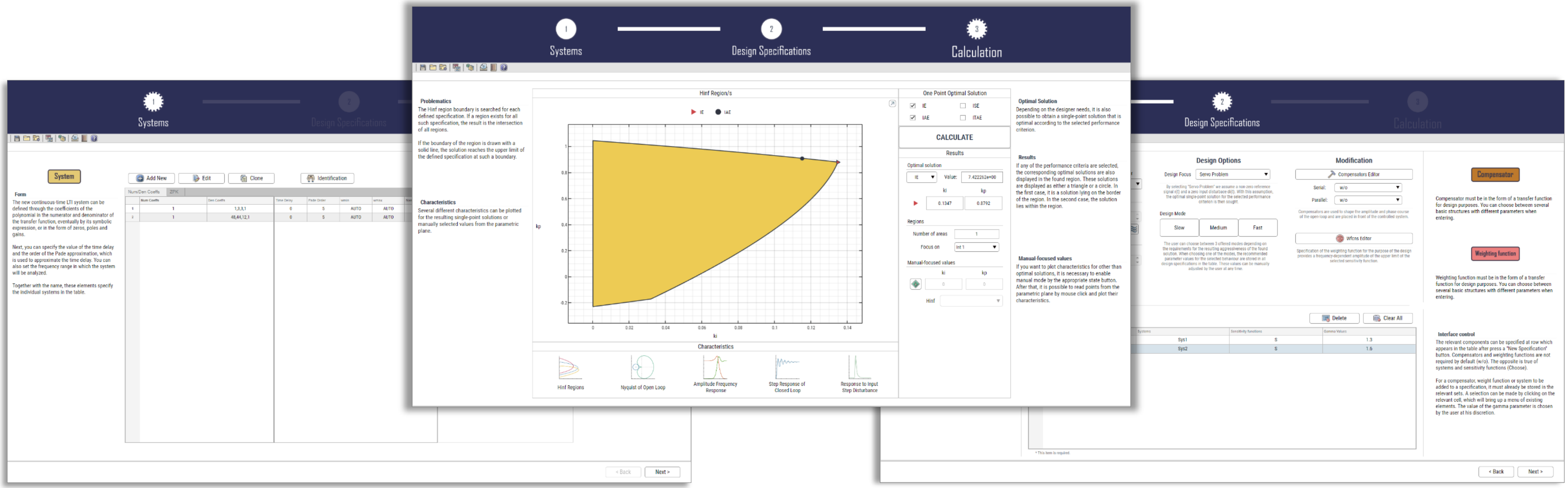
Multiple system models design or design with several H_∞ limitations can be implemented through the intersection of H_∞ regions

PID H_∞ Designer

www.pidlab.com

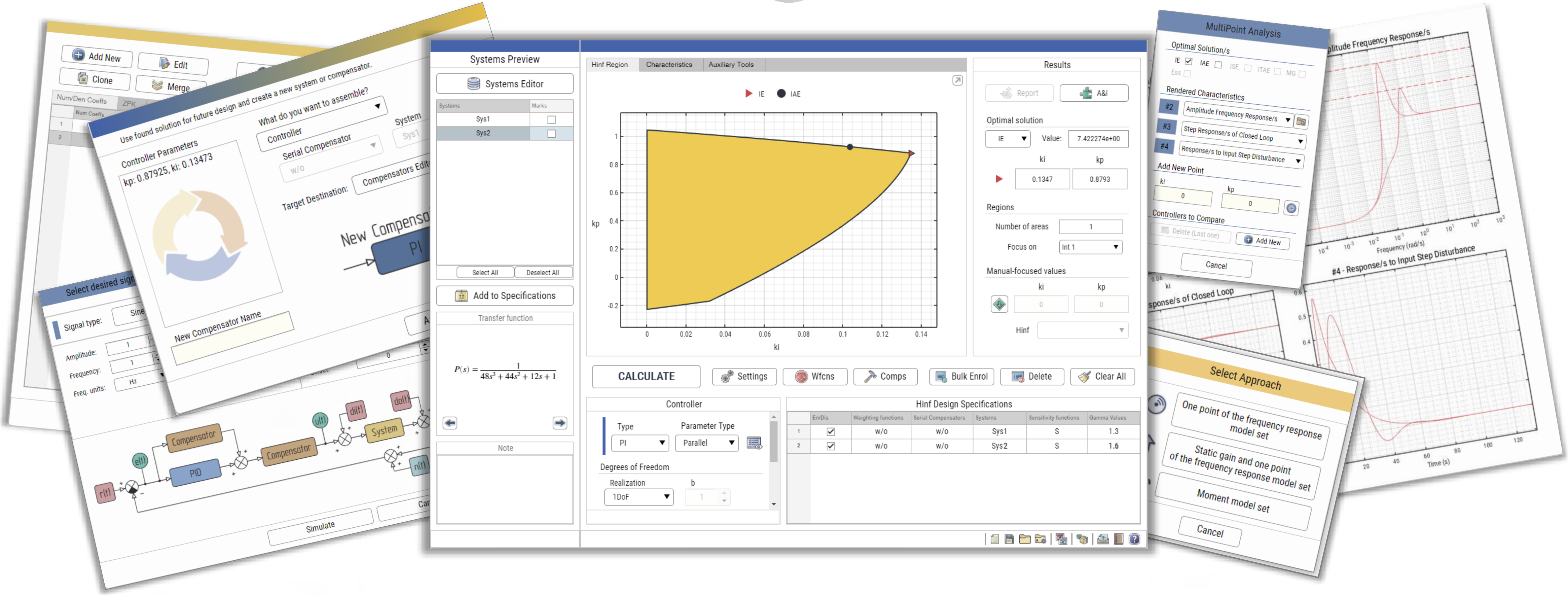
PID H_∞ Designer

StepByStep

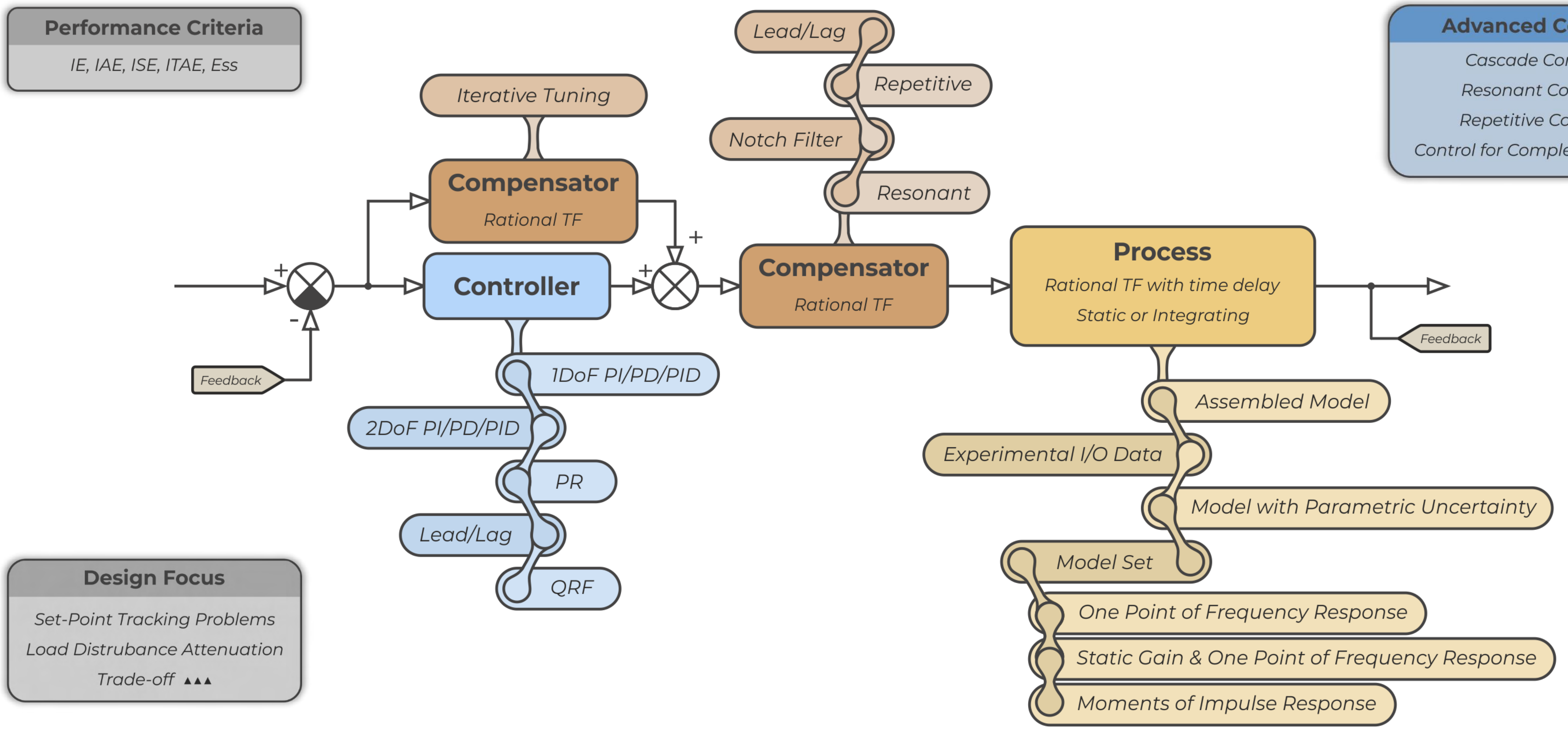


PID H_∞ Designer

WorkSpace



PID H_∞ Designer: Options



Examples

Design Specification of Robust PI-controller for Process Model Set

Proces model set: $$\mathbf{P} \triangleq \left\{ P_1(s) = \frac{-0.0216s + 0.0031}{s^2 + 0.457s + 0.0868} e^{-0.166s},\ P_2(s) = \frac{-0.0174s + 0.0046}{s^2 + 0.5978s + 0.0445} e^{-0.166s} \right\}$$

Requirements

Controller : $$C_{PI}(s) = K\left(1 + \frac{1}{T_i s}\right) = k_p + \frac{k_i}{s}$$

Design focus : Load step disturbance rejection (regulator problem)

Formulation of the design problem

Sensitivity functions : $$S_i(s) = \frac{1}{1 + C_{PI}(s)P_i(s)},\quad i = 1, 2$$

Weighting functions : $$W_i(s) = 1,\ i = 1, 2$$

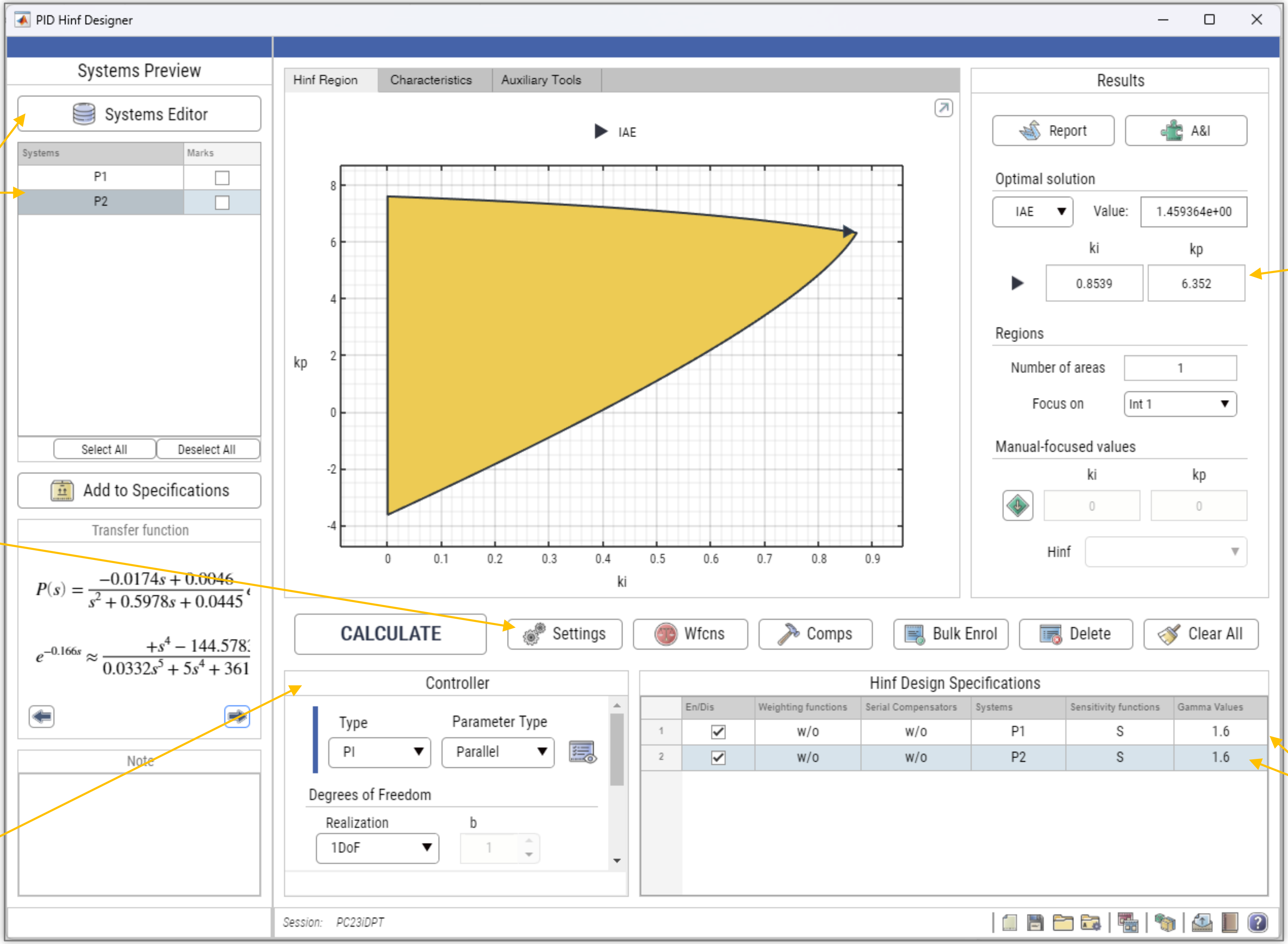
Design criterion : $$\min_{C_{PI}} \max_{i \in \{1,2\}} \int_0^\infty \left| e_i(t) \right| dt$$

PID
H_∞ Designer

Definition of processes transfer functions

Selection of the criterion function

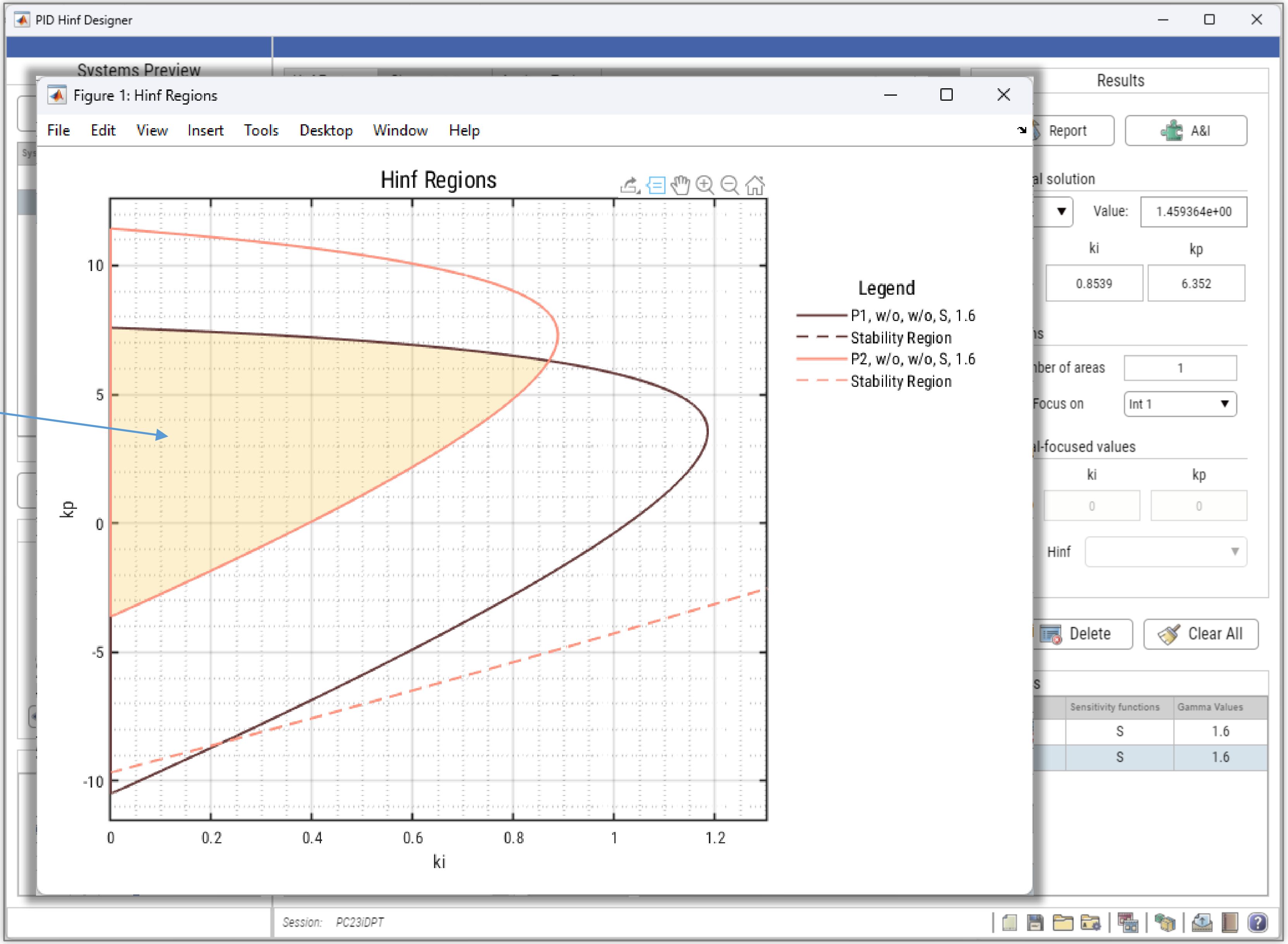
Setting the desired form of the controller



The resulting controller

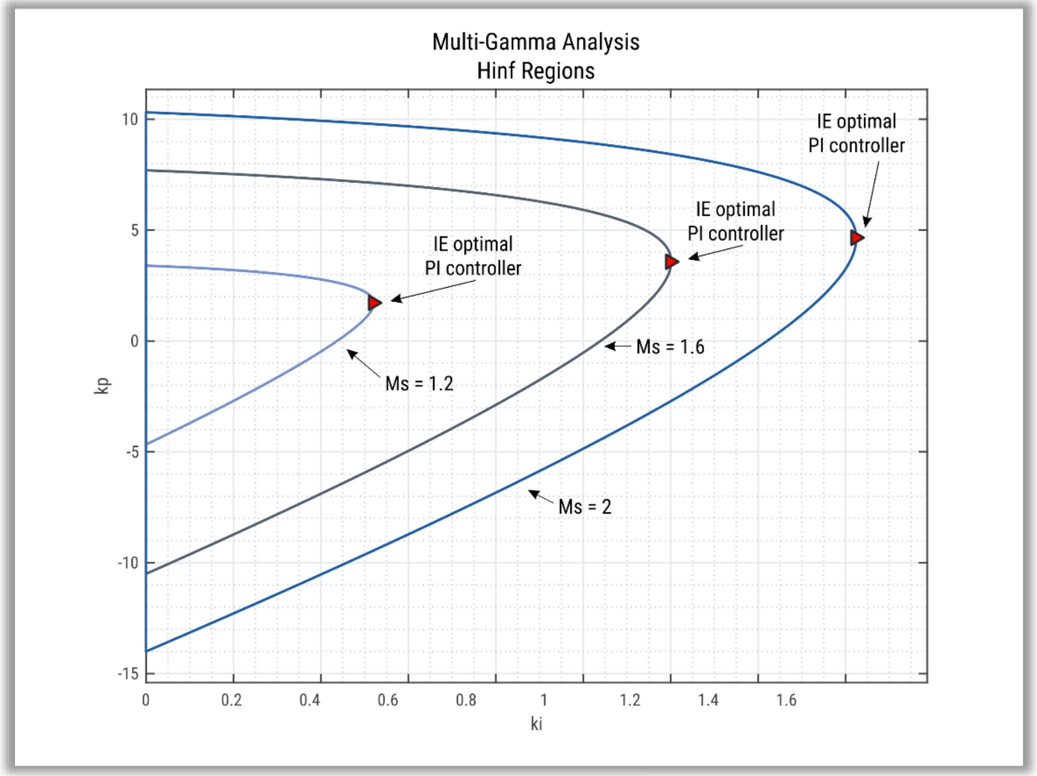
H_∞ limitations for S(s) function

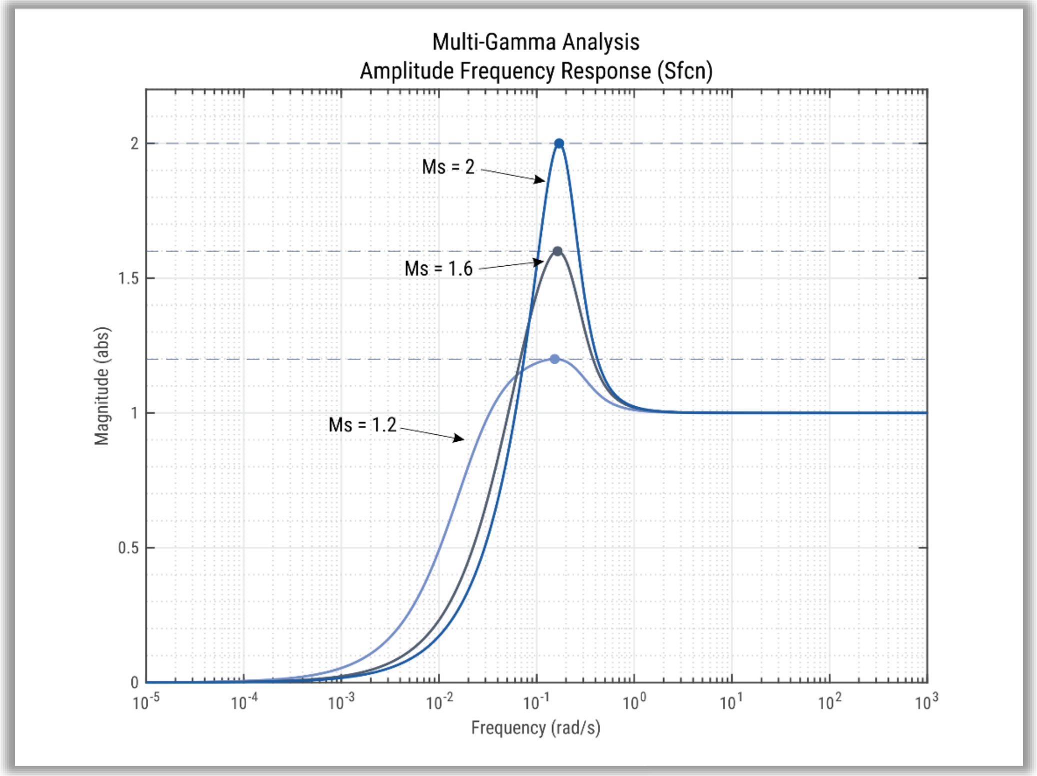
$$\left\|S(s)\right\|_\infty \le M_S$$



Intersection of H_∞ regions

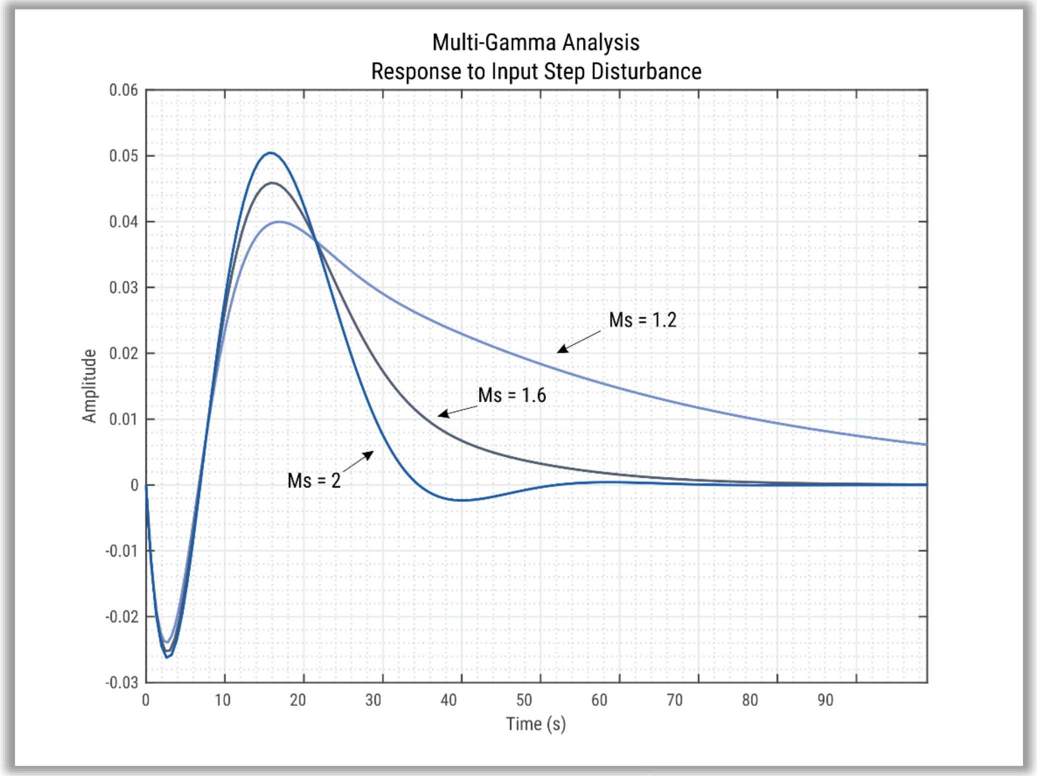
Selection of Gamma (Ms) Value

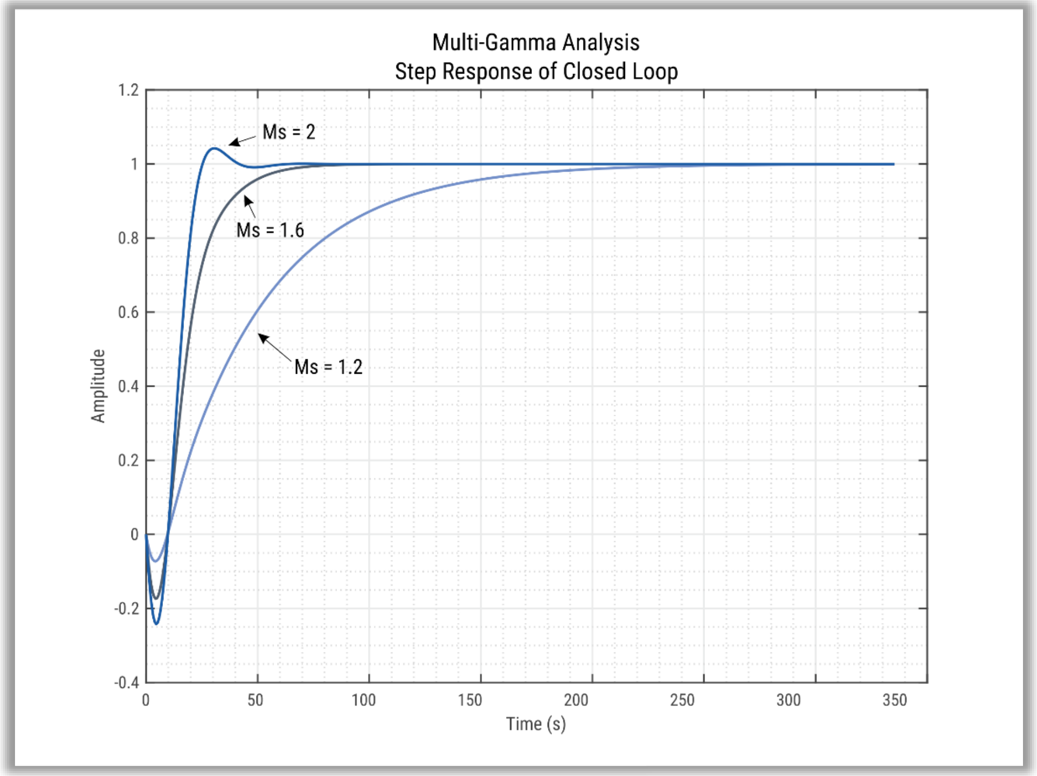




Amplitude frequency response of sensitivity function S(s) for IE optimal PI controllers

Selection of Gamma (Ms) Value





Grid Connected Photovoltaic Inverter

LCL filter with processing delay
for Single phase 3 kW Grid-Connected Photovoltaic Inverter system

$$P(s) = \frac{1}{1e-4s+1} \cdot \frac{s^2 + 1.143e4s + 1.587e8}{1.2e-3s^3 + 21.71s^2 + 3.016e5s}$$

Requirements

Controller: $C_{PR}(s) = k_p + k_r \frac{2\omega_c s}{s^2 + 2\omega_c s + \omega_0^2}$

Design focus: Track a current 50Hz sinusoidal reference waveform (servo problem)

Formulation of the design problem

Sensitivity functions: $S(s) = \frac{1}{1 + C_{PR}(s)P(s)}$

Weighting functions: $W(s) = 1$

Design parameters: $\omega_0 = 314.2 rad/s,\ \omega_c = 0.5 rad/s$

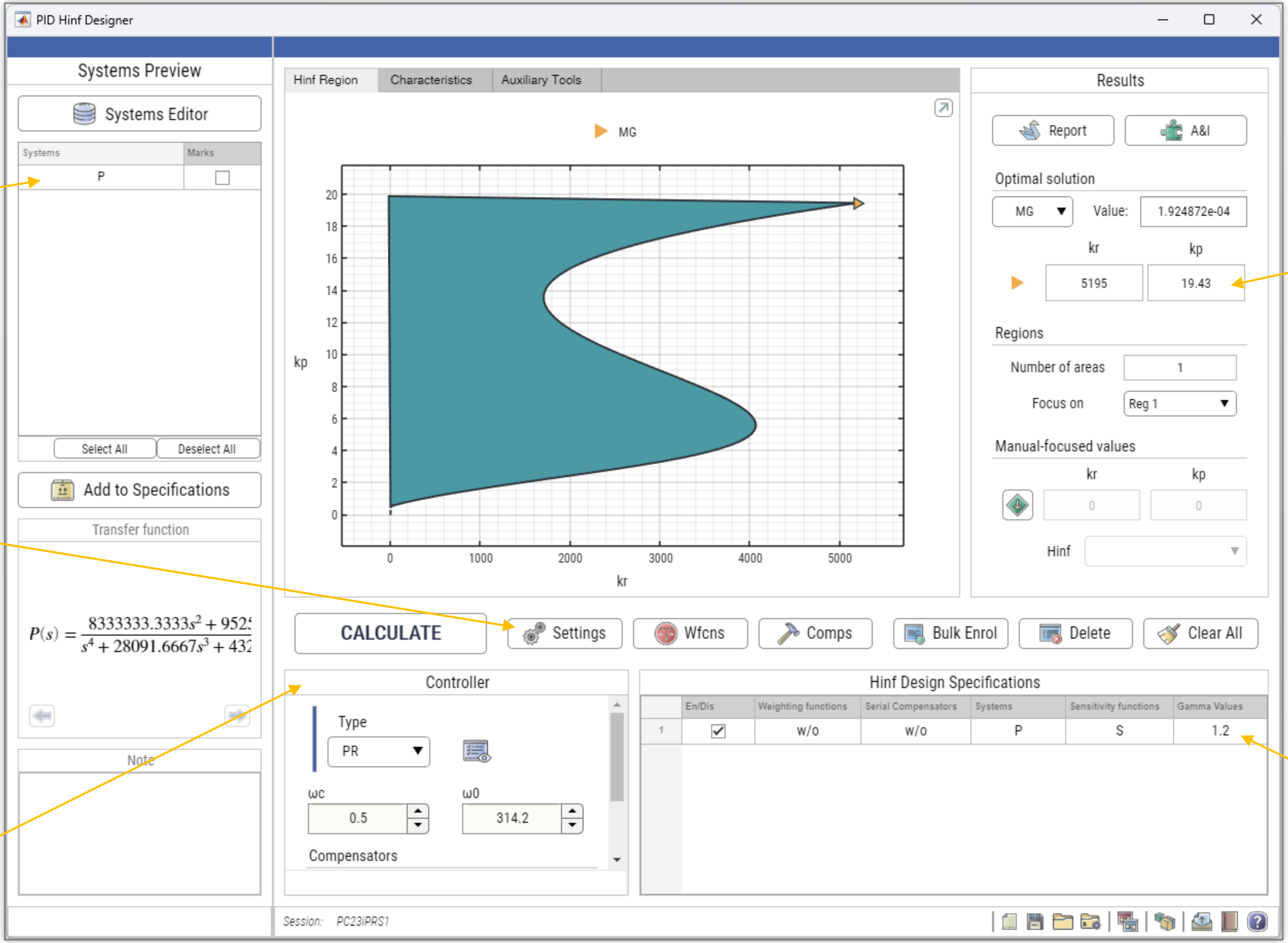
Design criterion : $\max_{C_{PR}} k_r$

PID
H_∞ Designer

Definition of system transfer function

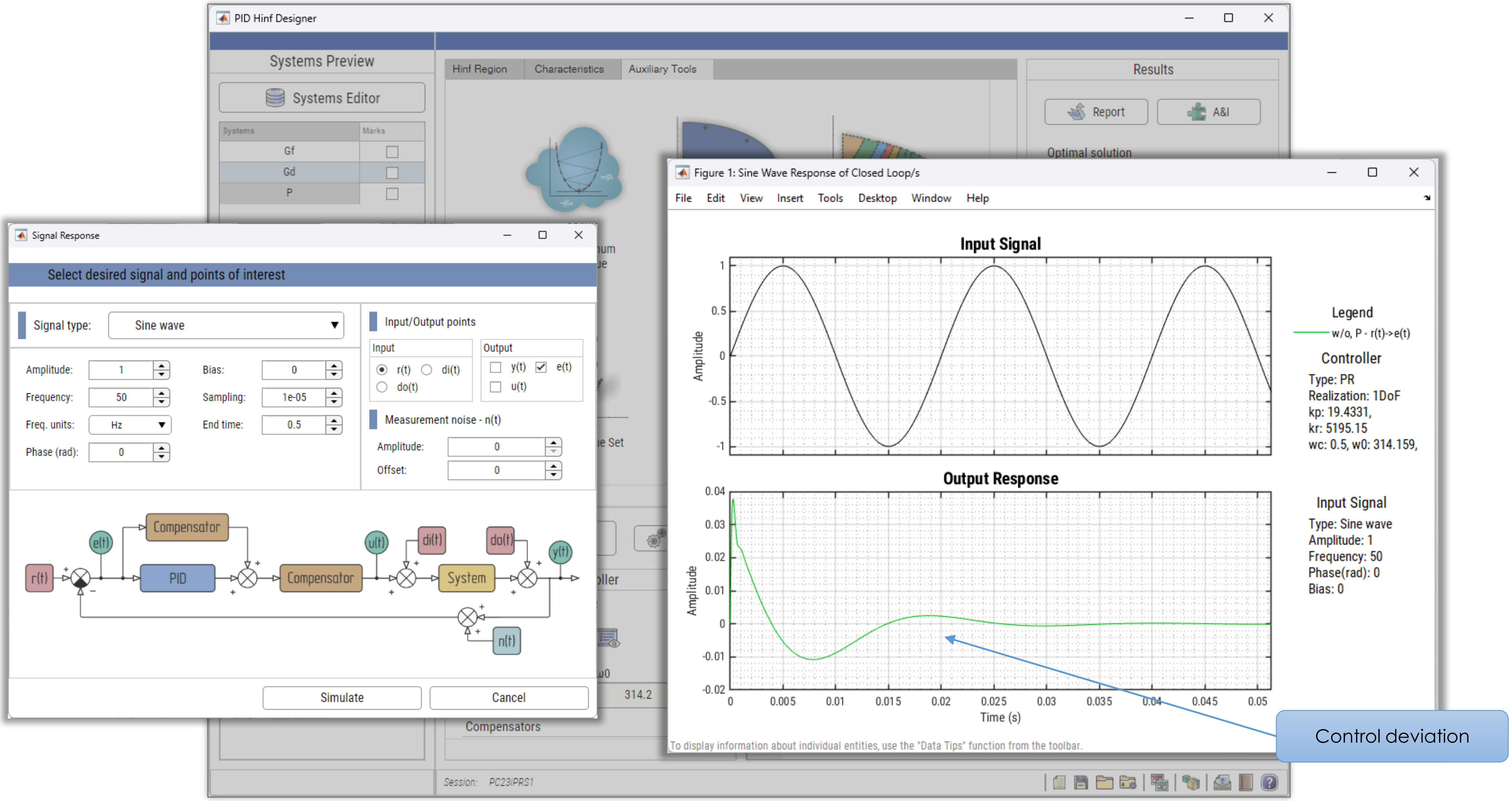
Selection of the criterion function

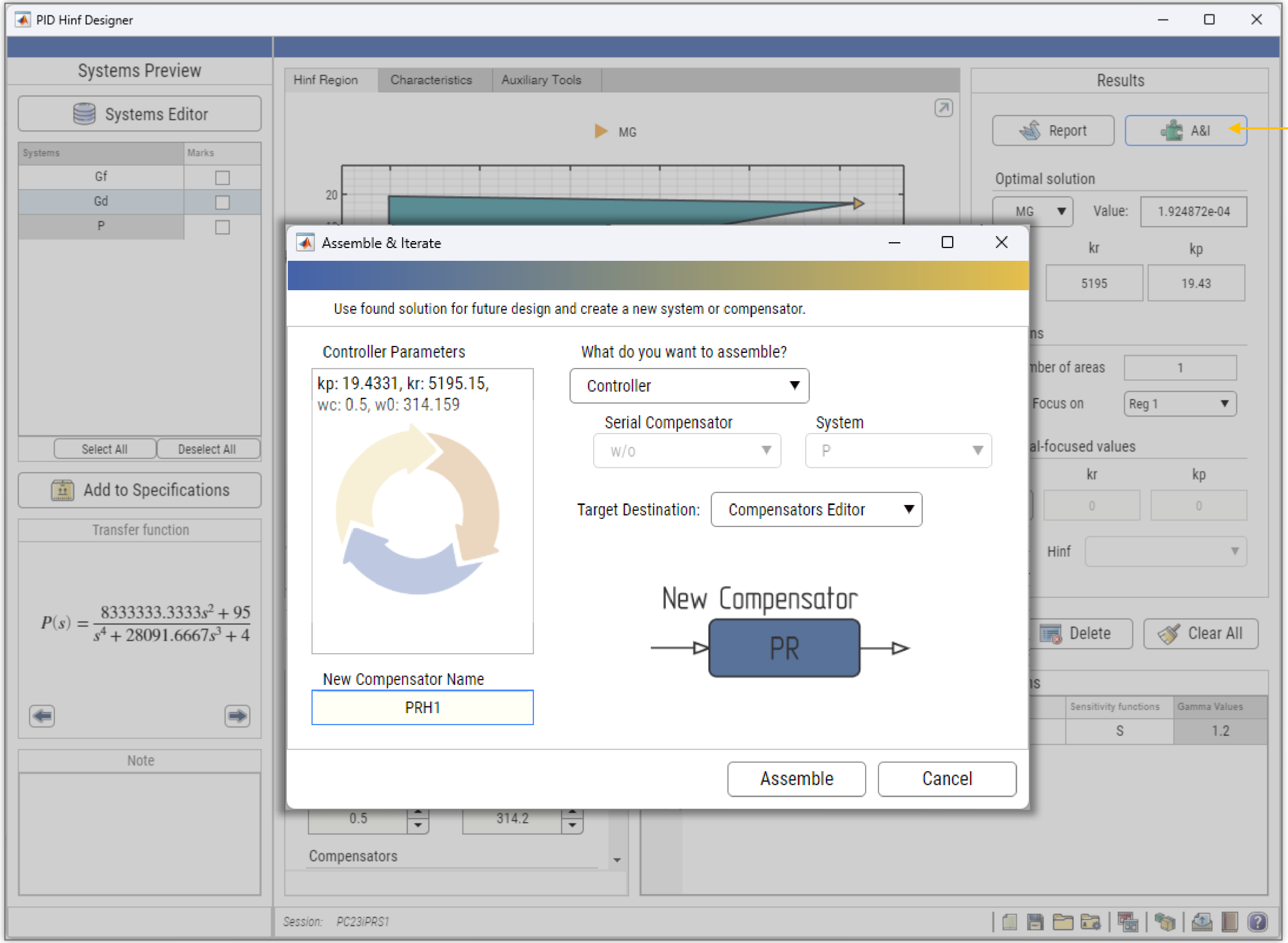
Setting the desired form of the controller



The resulting controller

H_∞ limitations for S(s) function

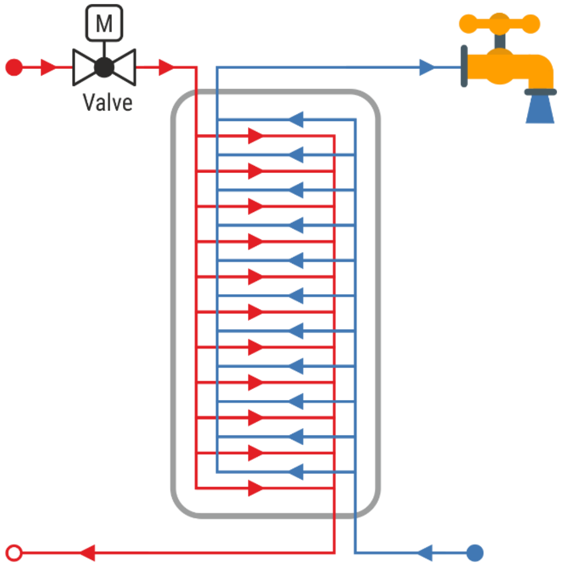

PID Hinf Designer
Systems Preview
Systems Editor
Hinf Region
Characteristics
Auxiliary Tools
Results
Report
A&I
Optimal solution
Signal Response
Select desired signal and points of interest
Signal type:
Sine wave
Amplitude: 1
Frequency: 50
Freq. units: Hz
Phase (rad): 0
Bias: 0
Sampling: 1e-05
End time: 0.5
Input/Output points
Input
r(t)
di(t)
do(t)
Output
y(t)
e(t)
u(t)
Measurement noise - n(t)
Amplitude: 0
Offset: 0
Compensator
PID
System
Simulate
Cancel
Figure 1: Sine Wave Response of Closed Loop/s
File Edit View Insert Tools Desktop Window Help
Input Signal
Amplitude
Output Response
Time (s)
Legend
w/o, P - r(t)->e(t)
Controller
Type: PR
Realization: 1DoF
kp: 19.4331,
kr: 5195.15
wc: 0.5, w0: 314.159,
Input Signal
Type: Sine wave
Amplitude: 1
Frequency: 50
Phase(rad): 0
Bias: 0
To display information about individual entities, use the "Data Tips" function from the toolbar.
Control deviation
Session: PC23IPRS1
Compensators
314.2



Iterative modification of the controller

Plate Heat Exchanger

Description of the process based on experimentally obtained Input/Output data



Requirements

Controller: $C_{PI}(s) = K\left(1 + \frac{1}{T_i s}\right) = k_p + \frac{k_i}{s}$

Design focus: Track reference temperature setpoint (servo problem)

Formulation of the design problem

Sensitivity functions: $S_i(s) = \frac{1}{1 + C_{PI}(s)P_i(s)}, \quad T_i(s) = \frac{C_{PI}(s)P_i(s)}{1 + C_{PI}(s)P_i(s)}, \quad i = 1,2,3$

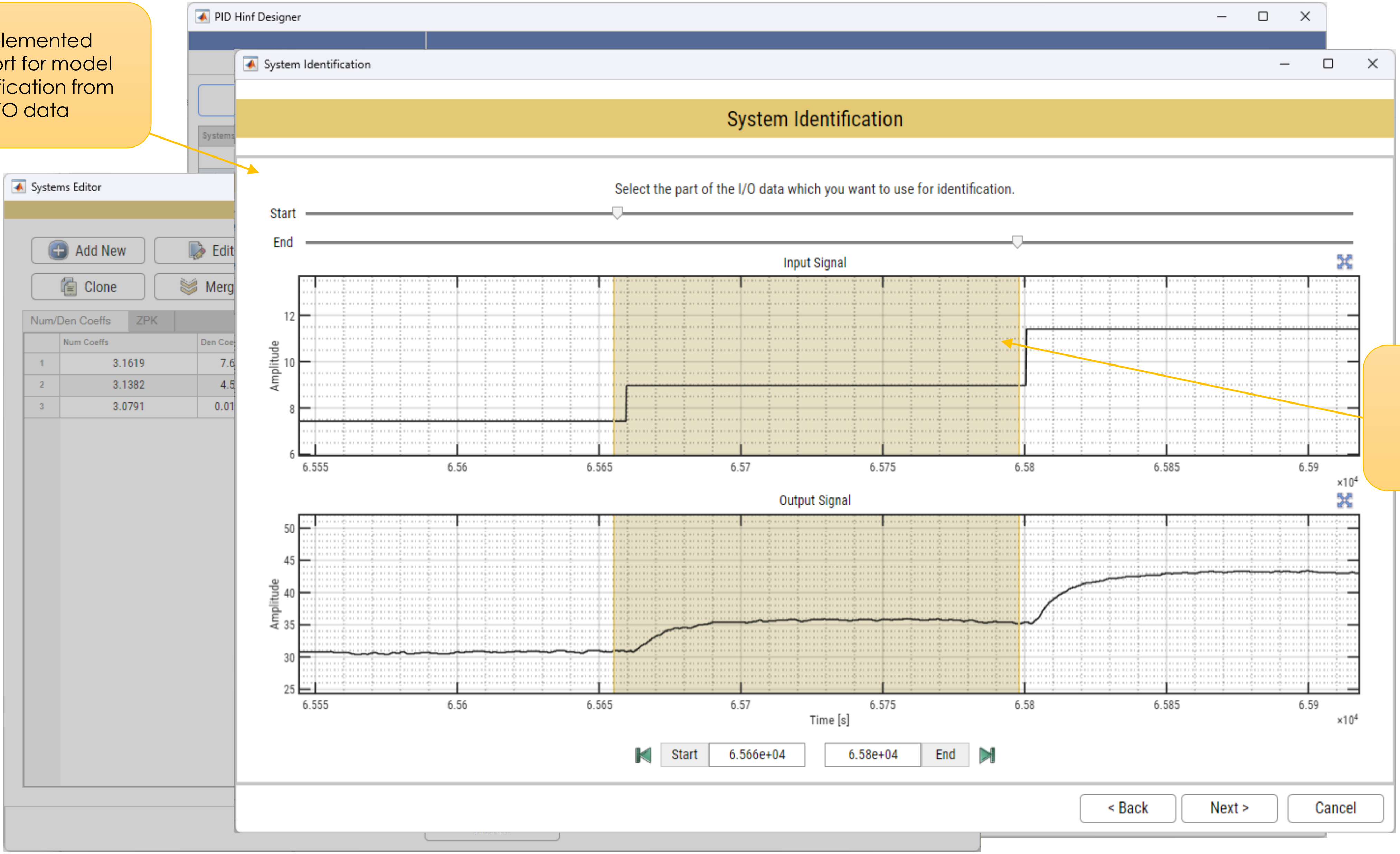
Weighting functions: $W_{S_i}(s) = 1, \quad W_{T_i}(s) = 1, \quad i = 1,2,3$

Design criterion: $\min_{C_{PI}} \max_{i \in \{1,2,3\}} \int_0^\infty |e_i(t)|\, dt, \quad \min_{C_{PI}} \max_{i \in \{1,2,3\}} \int_0^\infty t\,|e_i(t)|\, dt$

PID

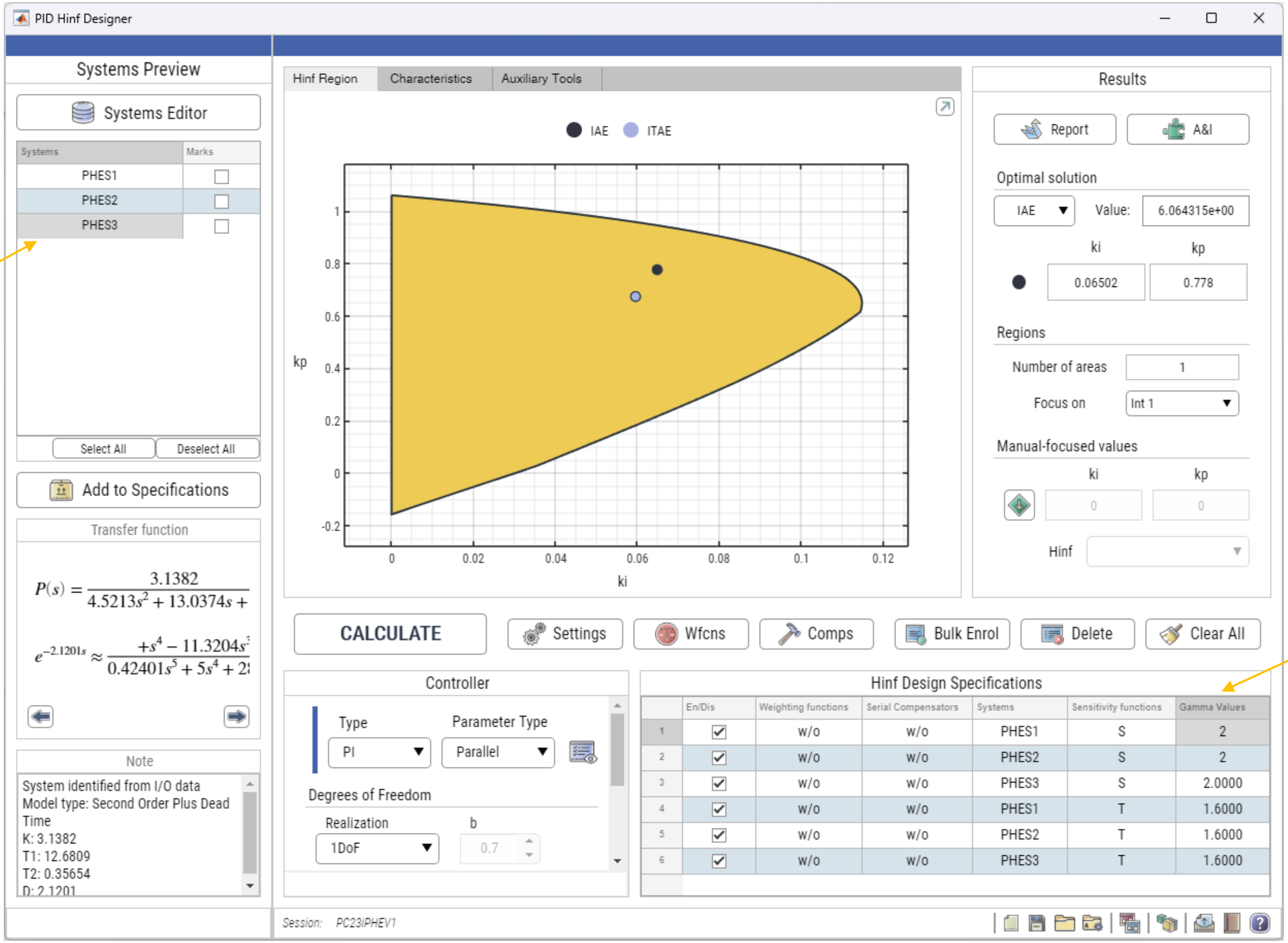
H_∞ Designer

Implemented support for model identification from I/O data



Identification of the model at different operating points of the process

Definition of identified models



H_∞ limitations for S(s) and T(s) functions

Conclusion

- New analytical method for the design of the H_∞ affine controller

- Affine controller includes almost all fixed structure controllers commonly used in practice

$$C(s,\mathbf{k}) \triangleq k_q Q(s) + k_r R(s) + F(s), \ \mathbf{k} \triangleq \left[k_r, k_q\right]^T \in \mathbb{R}^2$$

- Web tool PID H_∞ Designer (www.pidlab.com)