

# Analytical Design of a Wide Class of Controllers with Two Tunable Parameters Based on $H_\infty$ Specifications

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# Motivation for the development of a new method for the $H_\infty$ design of controllers with a fixed structure

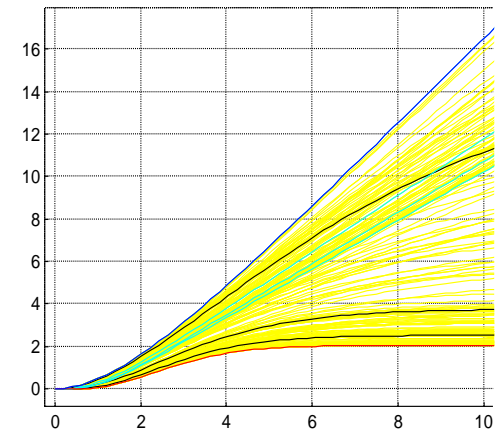
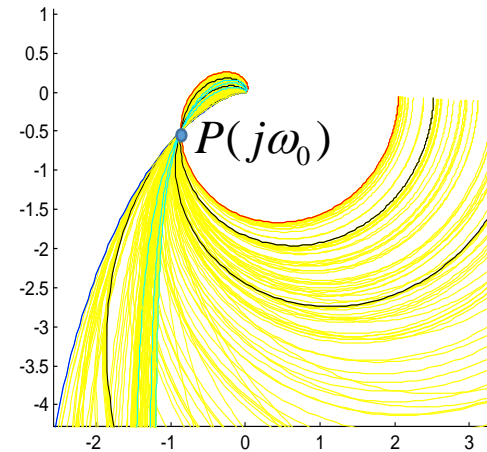
Design a fixed PI controller for a linear stable system with the transfer function in the form

$$P(s) = \frac{k_P}{s^l (\tau_1 s + 1)(\tau_2 s + 1) \cdots (\tau_N s + 1)}, \quad l \in \{0, 1\}, \quad \tau_i > 0, \quad N \text{ is arbitrary integer},$$

based on the knowledge of **only the single point** of the frequency response  $P(j\omega)$  at frequency  $\omega_0$ , that the sensitivity function of the closed loop satisfies the condition

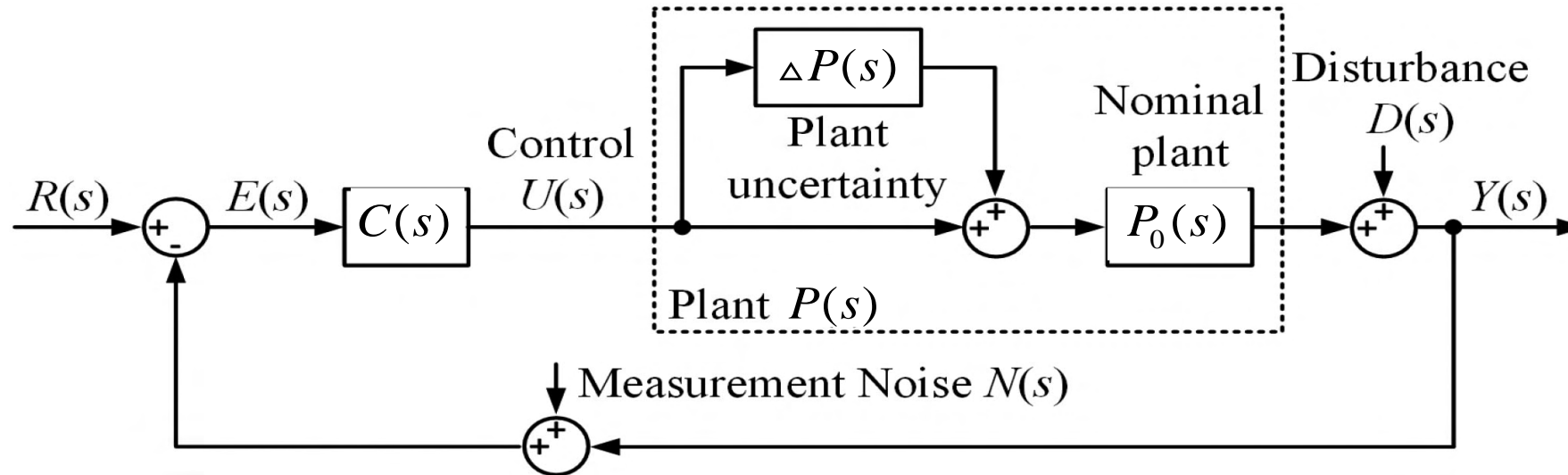
$$\forall \omega: |S(j\omega)| \triangleq \left| \frac{1}{1 + C_{PI}(j\omega)P(j\omega)} \right| \leq M_s$$
$$\Updownarrow$$
$$\|S(s)\|_\infty \leq M_s$$

for all transfer functions  $P(s)$  described above.



# A Typical Feedback Control Configuration

## Robustness/Performance Channels



$$Y(s) = T(s)(R(s) - N(s)) + S(s)D(s)$$

$$E(s) = S(s)(R(s) - D(s)) + T(s)N(s)$$

The Sensitivity Function

$$S(s) = \frac{1}{1 + C(s)P(s)},$$

The Complementary Sensitivity Function

$$T(s) = \frac{C(s)P(s)}{1 + C(s)P(s)},$$

The Input Sensitivity Function

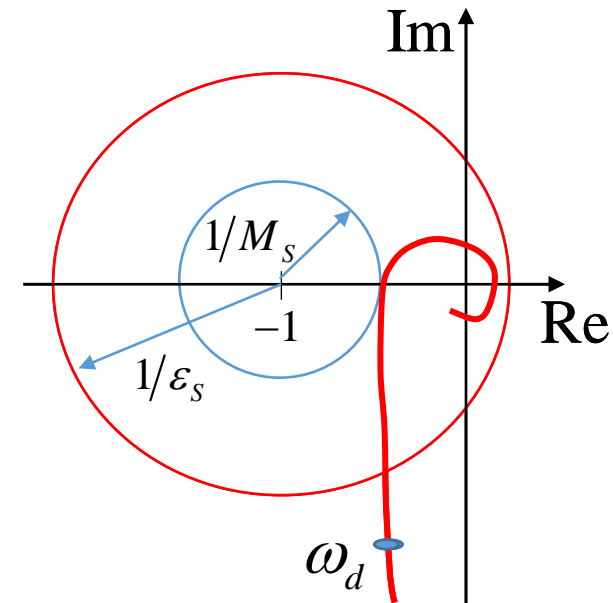
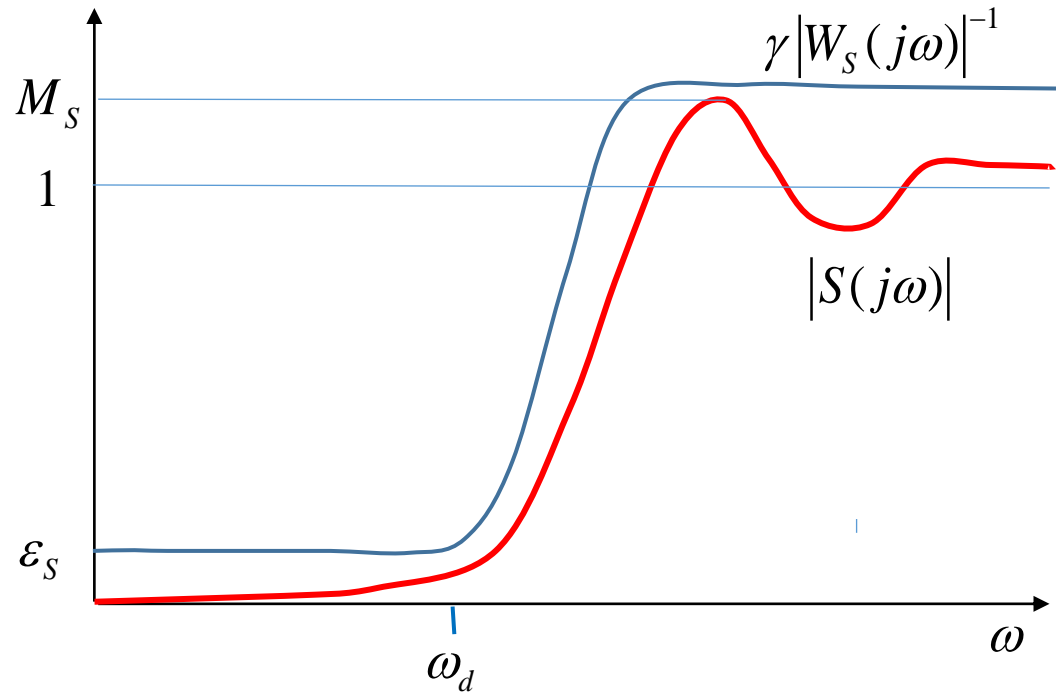
$$S_P(s) = \frac{P(s)}{1 + C(s)P(s)},$$

The Output Sensitivity Function

$$S_C(s) = \frac{C(s)}{1 + C(s)P(s)}$$

# Typical Constraint on Sensitivity Function

$$\|W_S(s)S(s)\|_\infty < \gamma \Leftrightarrow \forall \omega: |S(j\omega)| < \gamma |W_S(j\omega)|^{-1}$$



# *Some History of The $H_\infty$ Control Problem*

- The origins of the  $H_\infty$  -problem date back to the 1960s, when Zames discovered the small gain theorem and extended classical control techniques to MIMO control architectures.
- In nominal  $H_\infty$  synthesis, full-order  $H_\infty$  feedback controllers are computed via semidefined programming (1994) or algebraic Riccati equations (1989).
- Analytical  $H_\infty$  design of PI controllers (1998).
- Structured  $H_\infty$  synthesis in MATLAB (hinfstruct) via nonsmooth optimizers (2011).

# The Affine-structured Controller (AF-controller)

$$C_{AF}(s, k) \triangleq k_q Q(s) + k_r R(s) + F(s),$$

where  $Q(s)$ ,  $R(s)$ , and  $F(s)$  are arbitrary proper rational transfer functions;  $k_q$ , and  $k_r$  are tunable controller parameters, and  $k = [k_r, k_q]^T \in \mathbb{R}^2$  is the vector parameter.

## Typical examples :

PI controller:  $Q(s) = 1$ ,  $R(s) = \frac{1}{s}$ ,  $F(s) = 0$ ;

$$C_{PI}(s) = k_q + k_r \frac{1}{s}$$

PD controller:  $Q(s) = 1$ ,  $R(s) = \frac{s}{\tau s + 1}$ ,  $F(s) = 0$ ;

$$C_{PD}(s) = k_q + k_r \frac{s}{\tau s + 1}$$

PR controller:  $Q(s) = 1$ ,  $R(s) = \frac{2\omega_c s}{s^2 + 2\omega_c s + \omega_0^2}$ ,  $F(s) = 0$ ;

$$C_{PR}(s) = k_q + \frac{2\omega_c s k_r}{s^2 + 2\omega_c s + \omega_0^2}$$

PID controller:  $Q(s) = \frac{1}{s}$ ,  $R(s) = \frac{s}{\tau s + 1}$ ,  $F(s) = k_p$ ;

$$C_{PID}(s) = k_p + k_q \frac{1}{s} + k_r \frac{s}{\tau s + 1}$$

Other parameters except the two tunable ones must be fixed!

## Other Possible Choices of $Q(s)$ , $R(s)$ , and $F(s)$

- Converting the selected controller to an affine controller.

Example:  $C_{LL}(s) = k \cdot \frac{T_1 s + 1}{T_2 s + 1} \rightarrow C_{AF}(s) = k_q + k_r \frac{s}{T_2 s + 1}, Q(s) = 1, R(s) = \frac{s}{T_2 s + 1}$

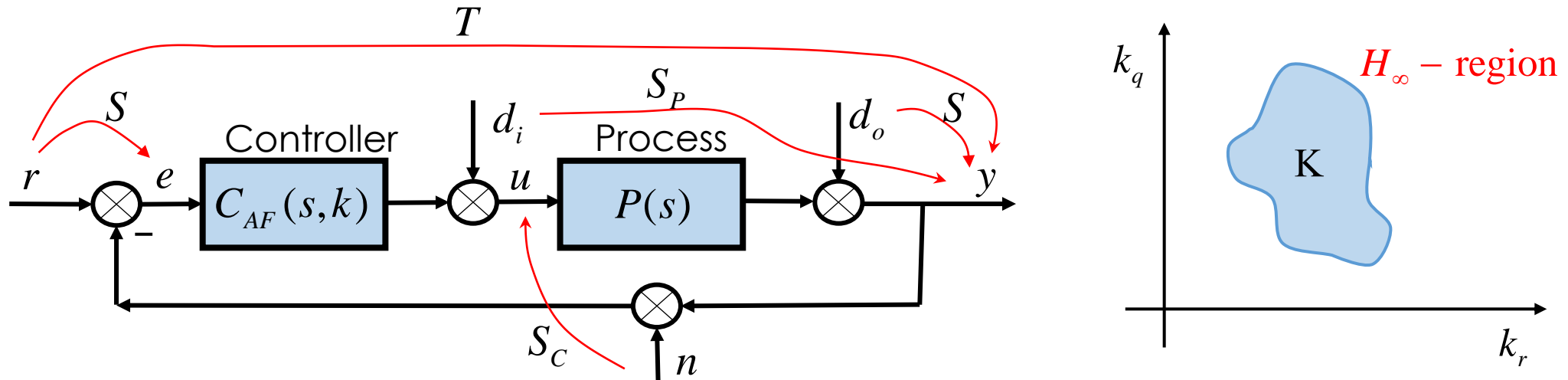
- Selection of  $Q(s)$ ,  $R(s)$ , and  $F(s)$  based on the Internal Model Principle.

Example: PR-controller (High gain of  $R(s) = \frac{2\omega_c s}{s^2 + 2\omega_c s + \omega_0^2}$  at the frequency  $\omega_0$ )

- Increasing the control performance by increasing the order of the existing controller  $C_0(s)$ .

Example:  $C_{new}(s) = C_0(s) + k_q Q(s) + k_r R(s) \rightarrow F(s) = C_0(s)$

# Basic $H_\infty$ Formulation of Affine-structured Controller Design Problem: Single $H_\infty$ Constraint



$H(s, k = [k_q, k_r]) = W(s)S_*(s, k)$ ,  $S_* \in \{S, T, S_C, S_P\}$  ..... weighted sensitivity function

$K \triangleq \left\{ [k_q, k_r] : \|H(s, k)\|_\infty \triangleq \sup_{\omega} |H(j\omega, k)| \leq \gamma, \text{ the closed-loop is internally stable} \right\}$  .....  $H_\infty$ -region in the parameter plane

1) Find the  $H_\infty$ -region  $K$  in the  $k_r$ - $k_q$  plane.

2) Find the optimal AF-controller in the  $H_\infty$ -region  $K$  with respect to some criterion, e.g.  $IAE \triangleq \int_0^\infty |e(t)| dt$  for the step in the reference value  $r$  (servo problem) or load disturbance  $d_i$  (regulator problem).

# D-decomposition for one $H_\infty$ Constraint $\|H(s, k)\|_\infty < \gamma$

$$\mathbf{K} \triangleq \{k : |H(j\omega, k)| < \gamma, \forall \omega \in [0, \infty)\} = \{k : |H_n(j\omega, k)| < \gamma |H_d(j\omega, k)|, \forall \omega \in [0, \infty)\}$$

**Theorem 1.** [\*] The boundary of the set  $\mathbf{K}$ , called the  $H_\infty$  region, is contained in the solution of the systems

Explicit solution for  
the case of AF-controller

$\begin{cases} H_n(j\omega, k) = 0, \\ H_d(j\omega, k) = 0, \end{cases}$	(1a)	Straight lines or empty set in the parametric plane $k_q - k_r$ .
	(1b)	

$\begin{cases}  H(j\omega, k) ^2 = \gamma^2, \\ \frac{d H(j\omega, k) ^2}{d\omega} = 0, \end{cases}$	(2a)	Critical curves $\varphi_k(\omega)$ , $k = 1, \dots, l, l \in \{0, 2, 4\}$ , in the parametric plane $k_q - k_r$ .
	(2b)	

for  $\omega \in [0, +\infty)$  and three equations

$ H(0, k)  = \gamma,$	(3)	Straight lines or empty set in the parametric plane $k_q - k_r$ .
$ H(\infty, k)  = \gamma,$	(4)	
$h_d(k) = 0,$	(5)	

where  $h_d$  is the coefficient at the higher order of a polynomial  $H_d(s, k)$ .

[\*] E. N. Gryazina, B. T. Polyak, A. A. Tremba, D-decomposition technique state-of-the-art, Automation and Remote Control, 2008.

## Explicit Solution of the system (2a), (2b) for the constraint

$$\|H(s,k)\|_{\infty} < \gamma, \quad H(s,k) \triangleq S(s,k)$$

Equation (2a)  $|H(j\omega, k)|^2 = \gamma^2$  is equivalent to  $p_1(\omega, k) = 0$ ,

Equation (2b)  $d|H(j\omega, k)|^2 / d\omega = 0$  is equivalent to  $p_2(\omega, k) = 0$ ,

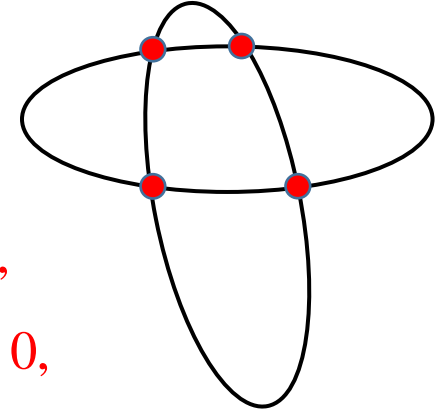
where  $p_1(\omega, k)$  and  $p_2(\omega, k)$  are second-order polynomials with real coefficients in the variables  $k_q$  and  $k_r$ .

**Proposition.** The system of polynomial equations

$$p_1(\omega, k) = a_1(\omega)k_q^2 + b_1(\omega)k_r^2 + c_1(\omega)k_qk_r + d_1(\omega)k_q + e_1(\omega)k_r + f_1(\omega) = 0,$$

$$p_2(\omega, k) = a_2(\omega)k_q^2 + b_2(\omega)k_r^2 + c_2(\omega)k_qk_r + d_2(\omega)k_q + e_2(\omega)k_r + f_2(\omega) = 0,$$

has analytical solutions in radicals. (The solution of this system can be converted to the solution of a **polynomial quartic equation** with one unknown.) There are two, four, or no real solutions to this system of equations. These solutions determine the parametric curves (critical curves  $\varphi_k$  from Theorem 1) with the parameter  $\omega$  in the parametric plane of the affine controller.



# Main Result: Isolation of $H_\infty$ - Region

The systems of equations (1), (2) and equations (3),(4) and (5) of Theorem 1 **have analytical solutions for affine controller  $C_{AF}(s)$  and the rational transfer function  $P(s)$ . These solutions define the critical curves and lines in the parametric plane that divide it into regions.**

Example:  $H_\infty$  – region for unstable process:

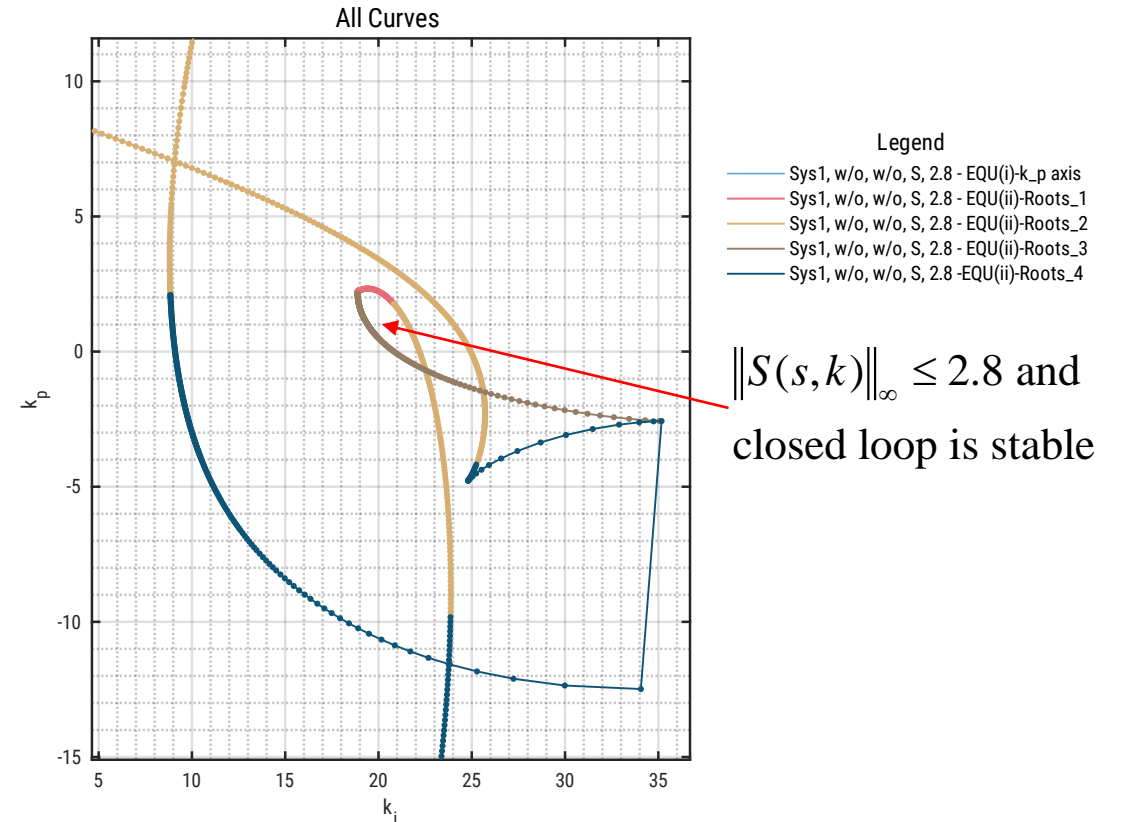
$$P(s) = \frac{s^3 + 4s^2 - s + 1}{s^5 + 2s^4 + 32s^3 + 14s^2 - 4s + 50},$$

$$C_{AF}(s, k) = C_{PI}(s, k) = k_p + \frac{k_i}{s},$$

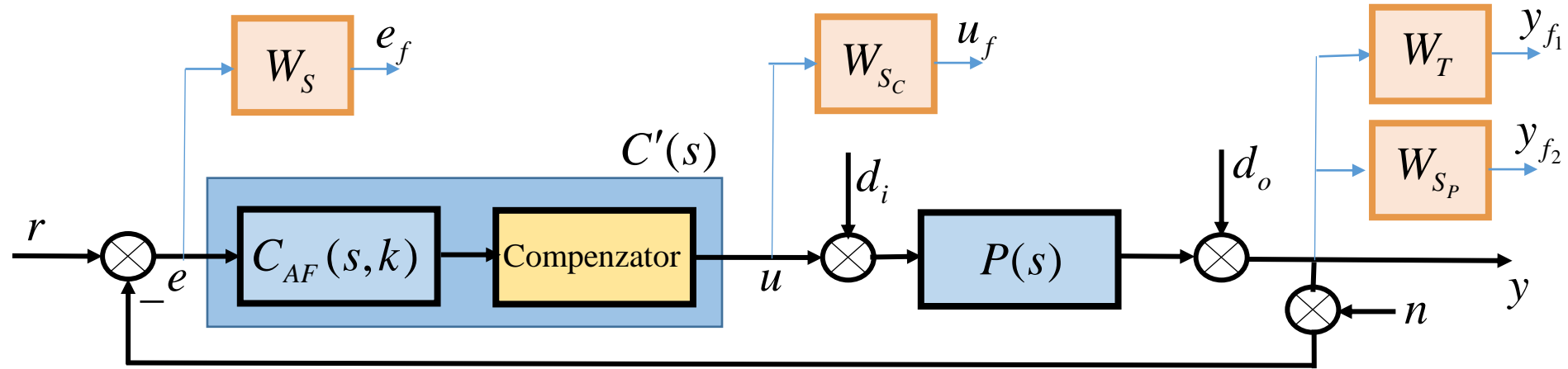
$$k_p = k_q, k_i = k_r,$$

$$Q(s) = 1, R(s) = \frac{1}{s}, F(s) = 0.$$

$$\|S(s, k)\|_\infty = \left\| \frac{1}{1 + C(s, k)P(s)} \right\|_\infty \leq M_S = 2.8$$



# $H_\infty$ Multiple Constraints



The performance or robustness channel

$$H_i \equiv H_i(P, C') = W_i S_{*i}, \quad i = 1, \dots, k$$

is a scalar weighted closed-loop sensitivity function.

Find all controllers  $C_{AF}$  for which it holds

$$\|H_i(P, C')\|_\infty \leq \gamma_i, \quad i = 1, \dots, k$$

subject to  $C'$  stabilizes  $P$  internally

$$C' \triangleq C_{AF}(s, k) \cdot C_{comp}.$$

# Multi-model Design Objectives

$H_\infty$  constraints on the robust/performance channels

The multi-model set:

$$\mathbf{P} \triangleq \{P_1, P_2, \dots, P_n\}$$

Design objectives:

- (i)  $\forall P \in \mathbf{P} : T(s) \text{ is stable} \wedge \|W_T(s)T(s)\|_\infty < \gamma_T,$
- (ii)  $\forall P \in \mathbf{P} : S(s) \text{ is stable} \wedge \|W_S(s)S(s)\|_\infty < \gamma_S,$
- (iii)  $\forall P \in \mathbf{P} : S_P(s) \text{ is stable} \wedge \|W_{S_P}(s)S_P(s)\|_\infty < \gamma_{S_P},$
- (iv)  $\forall P \in \mathbf{P} : S_C(s) \text{ is stable} \wedge \|W_{S_C}(s)S_C(s)\|_\infty < \gamma_{S_C},$

where  $W_S, W_T, W_{S_P}$ , and  $W_{S_C}$  are stable weighting functions,

$\|H\|_\infty \triangleq \sup_{\omega} |H(j\omega)|$  is called  $H_\infty$  - norm.

# Solving the Motivational Task

Input:

Model Set:

$$\omega_0 = 1,$$

$$P(j\omega_0) = e^{j*1.8},$$

$$N = 10,$$

$$P \triangleq \{P_1, P_2, \dots, P_{66}\},$$

$$P_i, i = 1, \dots, 66 \text{ samples}$$

of extremal systems\*

Design specification:

$$\text{PI-controller: } C_{PI}(s) = k_p + k_i \cdot 1/s$$

$$\forall P \in P: \|S(s, k)\| \leq 1.4,$$

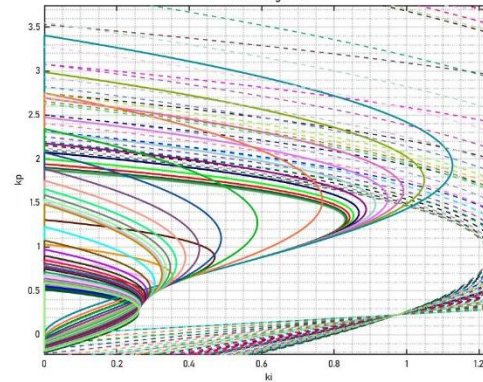
maximum integral gain

Output:

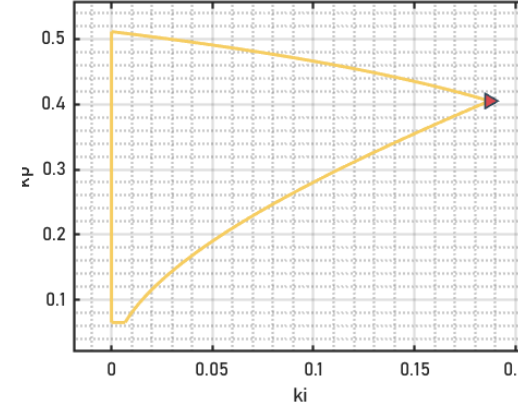
$$k_p = 0.4725$$

$$k_i = 0.3109$$

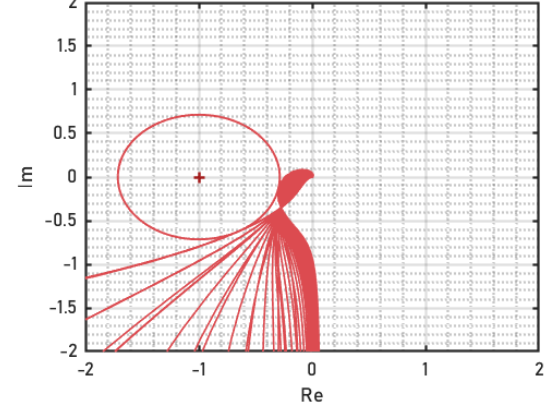
The intersection of all  $H_\infty$  regions



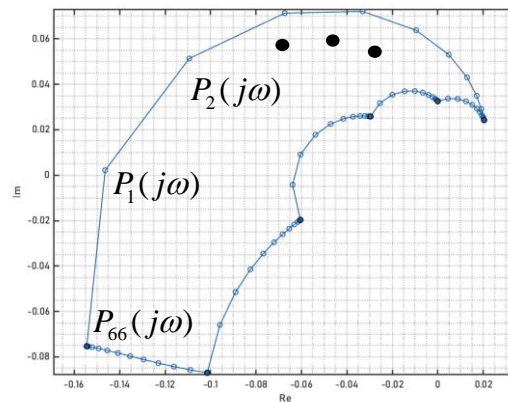
Hinf Region/s



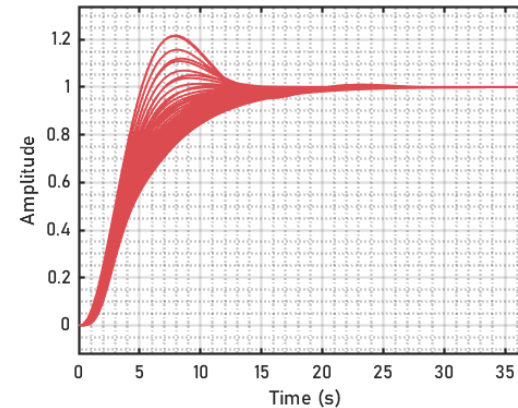
#2 - Nyquist/s of Open Loop



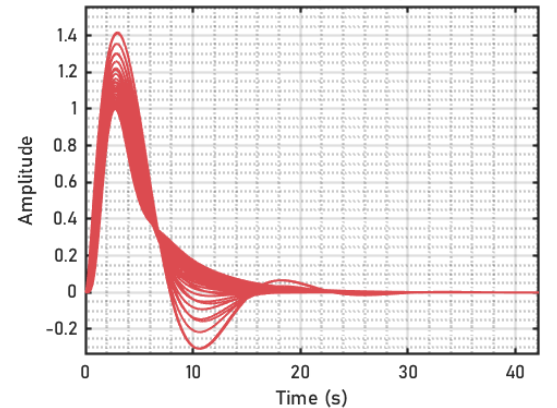
The boundary samples of the process value set



#3 - Step Response/s of Closed Loop



#4 - Response/s to Input Step Disturbance

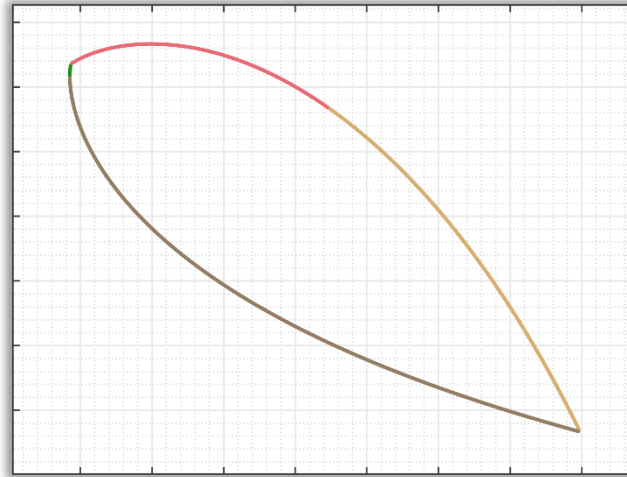
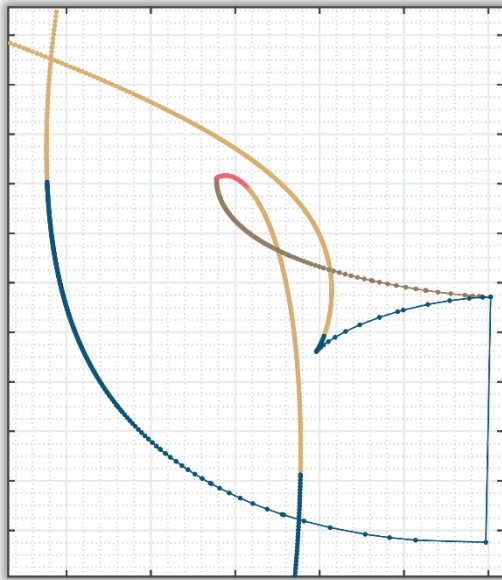


# Sketch of $H_\infty$ region isolation algorithm

# Single Specification Problem

## STEP 1

Finding all critical points/curves in the parametric plane

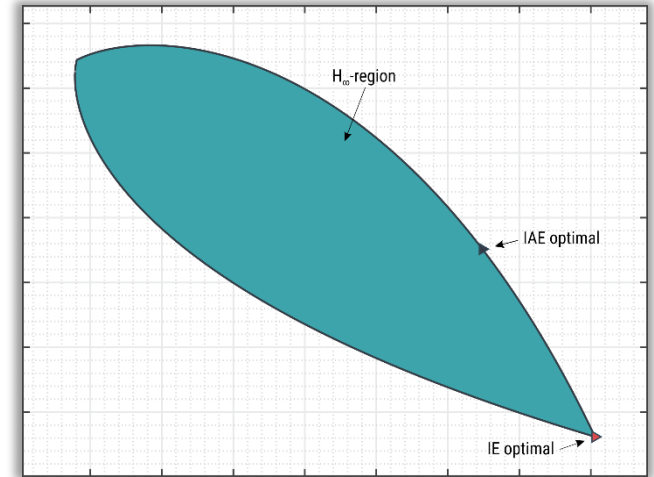


## STEP 2

Select parts that meet Hinf specifications and the condition of internal stability

### STEP 3

Create a border of region in  
a parametric plane

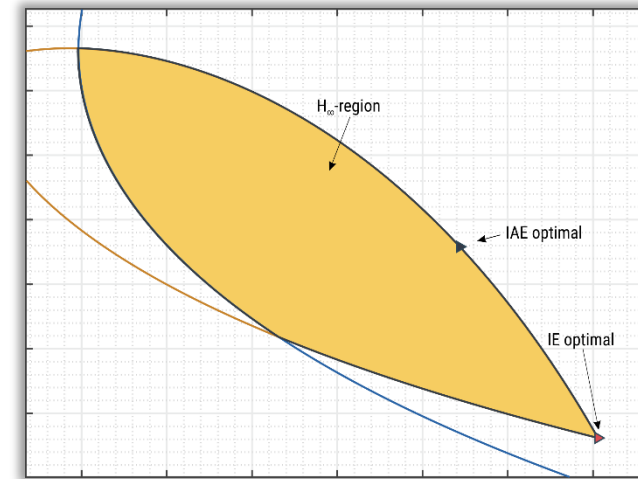
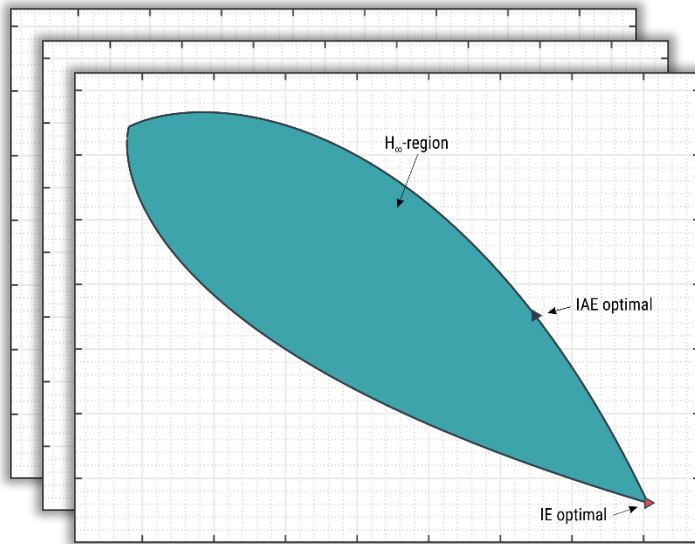


# Sketch of $H_\infty$ region isolation algorithm

## Multiple Specification Problem

### STEP 3i

Create a border of regions in a parametric plane for all specifications



### STEP 4

Multiple system models design or design with several  $H_\infty$  limitations can be implemented through the intersection of  $H_\infty$  regions

# PID $H_{\infty}$ Designer

[www.pidlab.com](http://www.pidlab.com)

Three overlapping screenshots of the PID Hinf Designer software interface. The central screenshot shows the main title screen with the text "PID Hinf Designer" and a large 3D hexagonal graphic. The left screenshot shows the "Systems Editor" and "Transfer function" input area. The right screenshot shows the "Calculation" results page with various parameters and a plot.

**Version: 3.2.0**

Home Approach UI

# PID<sub>H<sub>inf</sub></sub> Designer

The design of the controller and control loop is a crucial aspect of automation, which is aimed at managing diverse technological processes and equipment. The proper configuration of these control elements significantly impacts the efficiency and effectiveness of the resulting system. The purpose of this application is to find such a suitable configuration.

**Get Started >>**

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**Systems Preview**

Systems Editor

Systems: Sys1, Sys2

Select All Deselect All

Add to Specifications

Transfer function

$$P(s) = \frac{1}{48s^3 + 44s^2 + 12s + 1}$$

Note

**Calculation**

One Point Optimal Solution

Optimal solution

Results

Optimal solution: IE Value: -7.448505e-02

ki: -13.39 kp: 0.7175

Regions

Number of areas: 1

Focus on: Reg 1

Manual-focused values

ki: 0 kp: 0

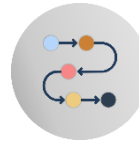
Hinf: 0

Manual-focused values

If you want to plot characteristics for other than optimal solutions, it is necessary to enable manual mode by the appropriate state button. After that, it is possible to read points from the parametric plane by mouse click and plot their characteristics.

# PID $H_{\infty}$ Designer

## StepByStep



**1 Systems** | **2 Design Specifications** | **3 Calculation**

**System**

The new continuous time LT1 system can be defined through the coefficients of the polynomial in the numerator and denominator of the transfer function, eventually by its symbolic expression, or in the form of zeros, poles and gains.

Next, you can specify the value of the time delay and the order of the Padé approximation, which is used to approximate the time delay. You can also set the frequency range in which the system will be analyzed.

Together with the name, these elements specify the individual systems in the table.

Num Den Coeffs	Time Delay	Padé Order	Order	Order
1 1	0	5	AUTO	AUTO
1 1	0	5	AUTO	AUTO

**Problematics**

The Hinf region boundary is searched for each defined specification. If a region exists for all such specification, the result is the intersection of all regions.

If the boundary of the region is drawn with a solid line, the solution reaches the upper limit of the defined specification at such a boundary.

**Characteristics**

Several different characteristics can be plotted for the resulting single-point solutions or manually selected values from the parametric plane.

**Hinf Region/s**

One Point Optimal Solution

**Results**

Optimal solution:  $7.422212e+90$

**Manual focused values**

**Design Options**

Design Focus: Servo Problem

Design Mode: Slow Medium Fast

**Modification**

Compensator Editor

Serial: w/o

Parallel: w/o

Weighting function editor

Specify the weighting function for the purpose of the design provides a frequency dependent amplitude of the upper limit of the selected sensitivity function.

**Interface control**

The relevant components can be specified at row which appears in the table after press a "New Specification" button. Compensators and weighting functions are not required by default (w/o). The opposite is true of systems and sensitivity functions (Choose).

For a compensator, weight function or system to be added to a specification, it must already be entered in the relevant sets. A selection can be made by clicking on the relevant cell, which will bring up a menu of existing elements. The value of the gamma parameter is chosen by the user at his discretion.

**Characteristics**

Hinf Regions

Nyquist of Open Loop

Amplitude Frequency Response

Step Response of Closed Loop

Response to Input Step Disturbance

**Back** **Next**

# PID H<sub>∞</sub> Designer

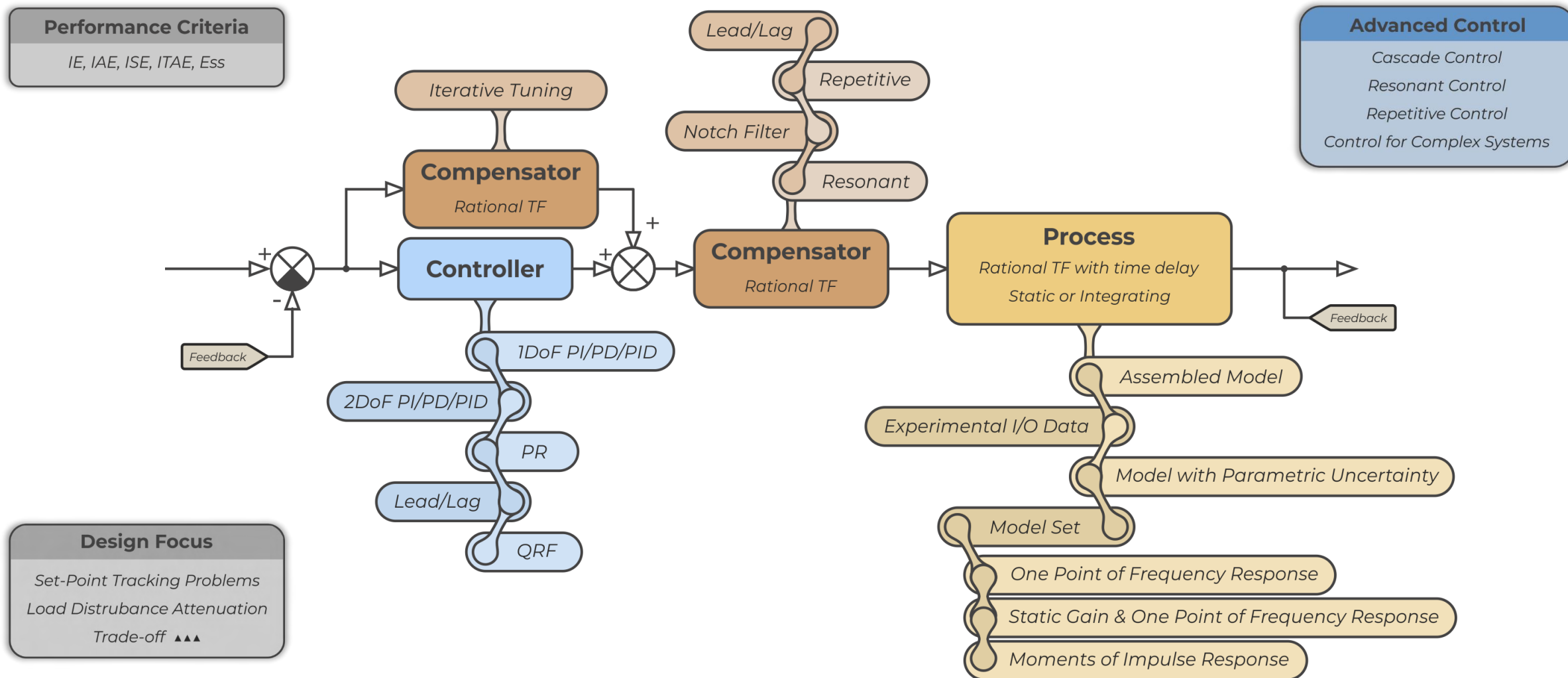
## WorkSpace



The collage displays several key features of the PID H<sub>∞</sub> Designer software:

- Compensator Design:** A window titled "Use found solution for future design and create a new system or compensator." showing controller parameters:  $k_p: 0.87925$ ,  $k_i: 0.13473$ . It includes a "New Compensator" button and a "Simulate" button.
- Systems Editor:** A table listing systems for design.
- Transfer function:** 
$$P(s) = \frac{1}{48s^3 + 44s^2 + 12s + 1}$$
- Controller Design:** A window showing the selected controller type (PI) and parameter type (Parallel). It includes a "Degrees of Freedom" section with "Realization" set to 1DoF and "b" set to 1.
- Hinf Design Specifications:** A table summarizing the design requirements.
- Results and Analysis:** A window showing the optimal solution (IE, Value: 7.422274e+00) and rendered characteristics, including a plot of the Amplitude Frequency Response/s.
- MultiPoint Analysis:** A window showing the optimal solution/s and rendered characteristics, including a plot of the Step Response/s of Closed Loop.
- Select Approach:** A window showing the selected approach: "One point of the frequency response model set".

# PID H<sub>∞</sub> Designer: Options



# Examples



# Design Specification of Robust PI-controller for Process Model Set

Proces model set:  $\mathbf{P} \triangleq \left\{ P_1(s) = \frac{-0.0216s + 0.0031}{s^2 + 0.457s + 0.0868} e^{-0.166s}, P_2(s) = \frac{-0.0174s + 0.0046}{s^2 + 0.5978s + 0.0445} e^{-0.166s} \right\}$

## Requirements

Controller :  $C_{PI}(s) = K \left( 1 + \frac{1}{T_i s} \right) = k_p + \frac{k_i}{s}$

Design focus : Load step disturbance rejection (regulator problem)

## Formulation of the design problem

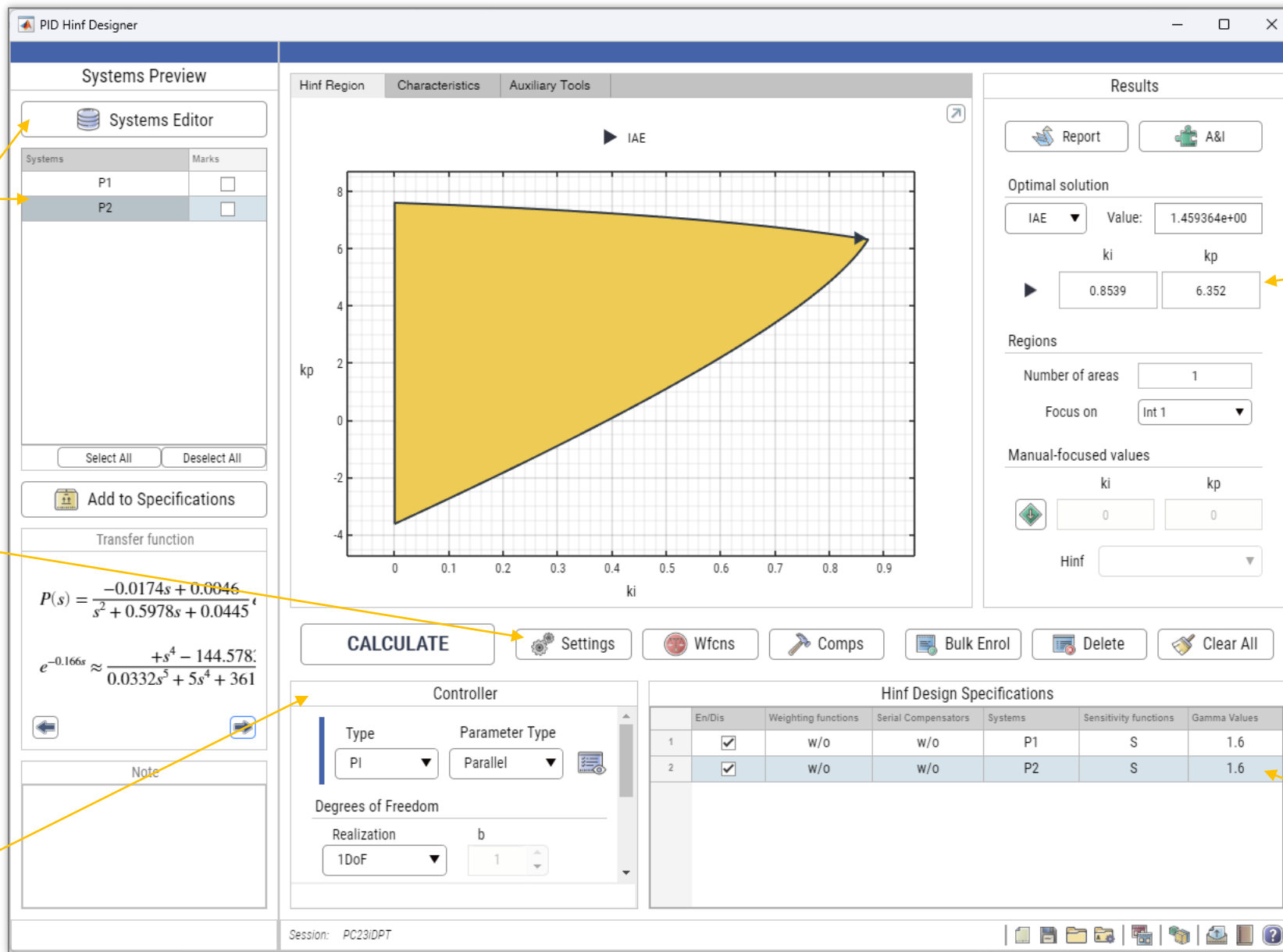
Sensitivity functions :  $S_i(s) = \frac{1}{1 + C_{PI}(s)P_i(s)}, i = 1, 2$

Weighting functions :  $W_i(s) = 1, i = 1, 2$

Design criterion :  $\min_{C_{PI}} \max_{i \in \{1, 2\}} \int_0^\infty |e_i(t)| dt$



**PID  
H<sub>∞</sub> Designer**



Definition of processes transfer functions

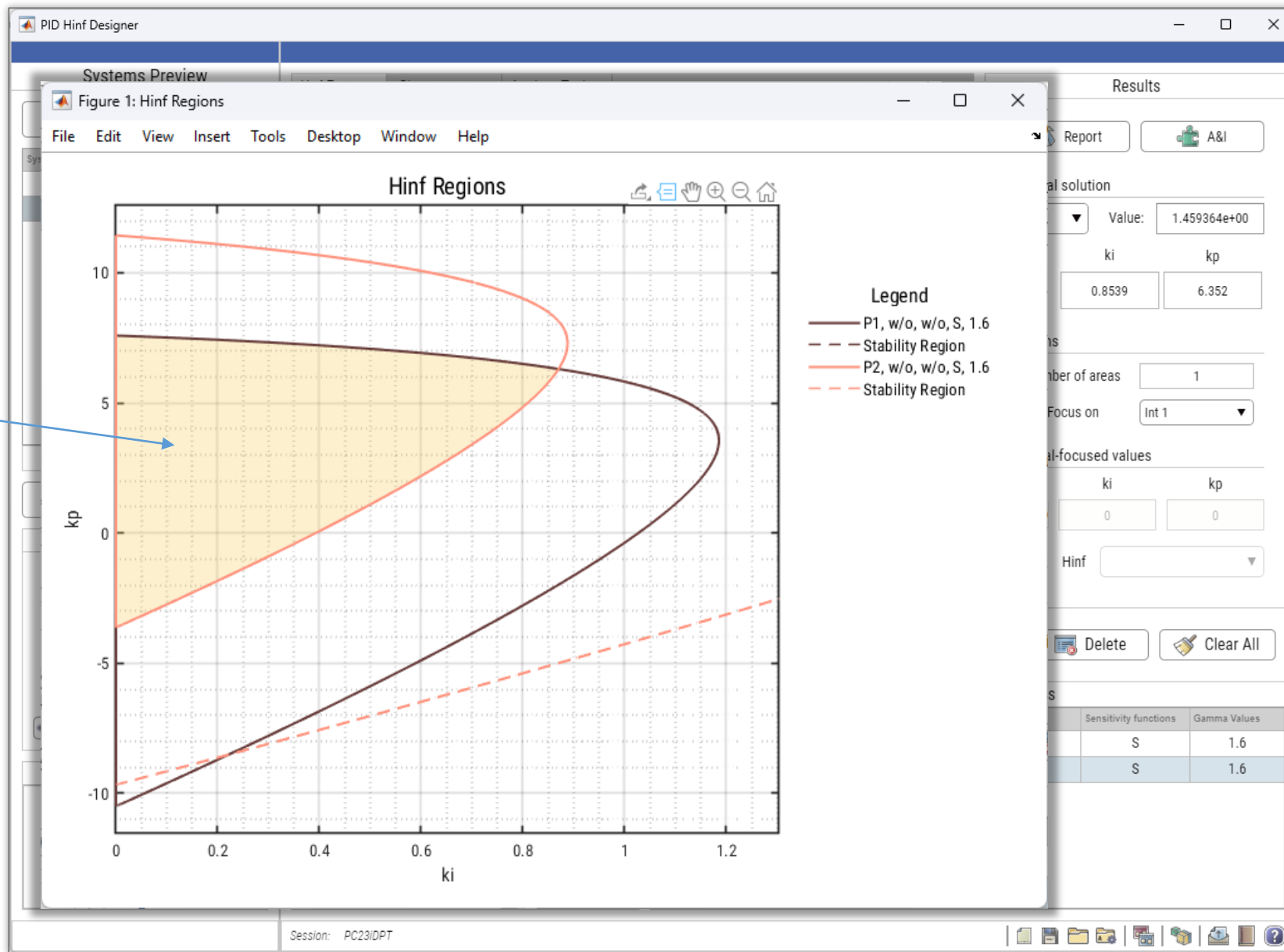
Selection of the criterion function

Setting the desired form of the controller

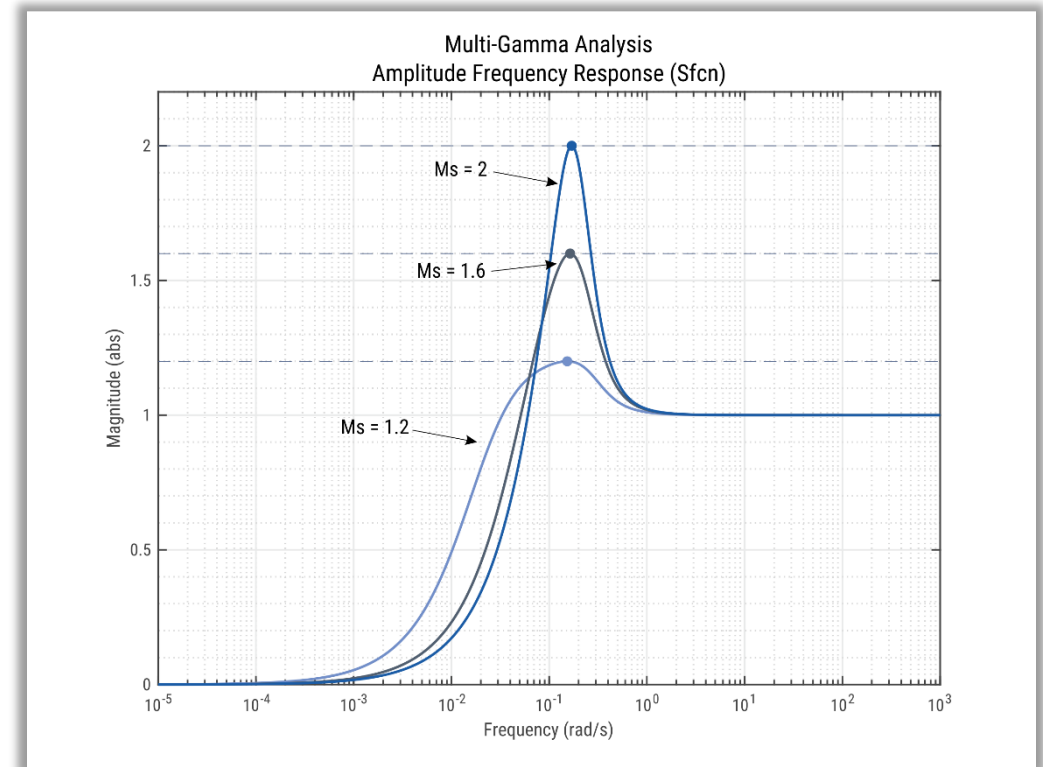
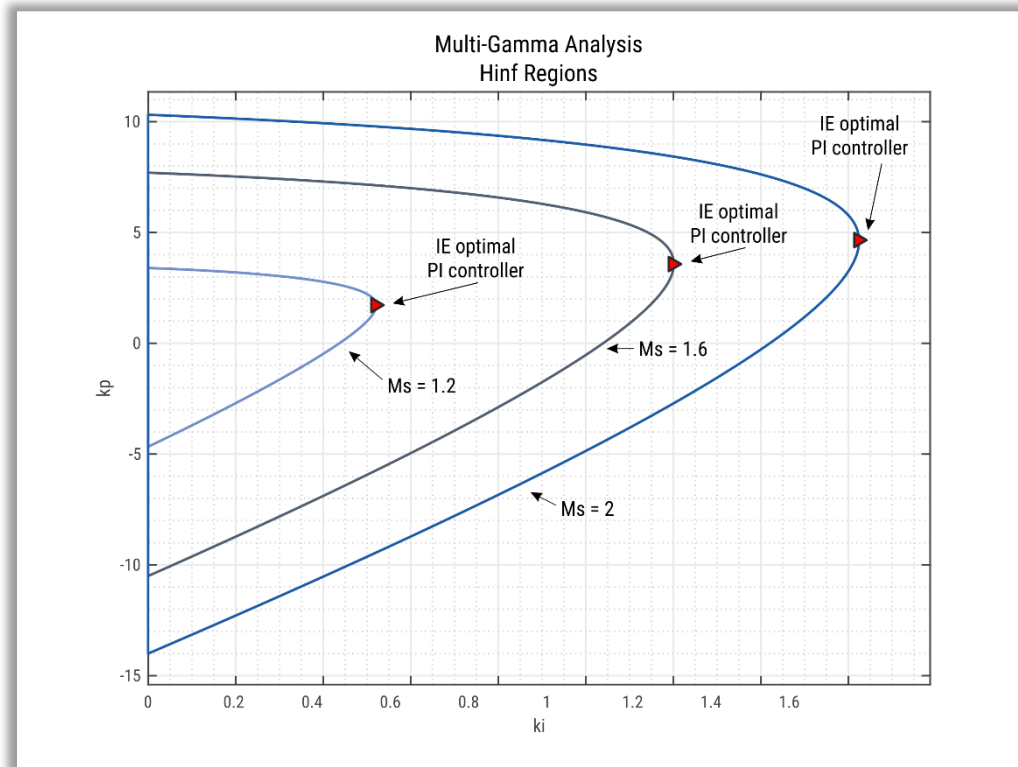
The resulting controller

$H_\infty$  limitations for  $S(s)$  function

$$\|S(s)\|_\infty \leq M_s$$

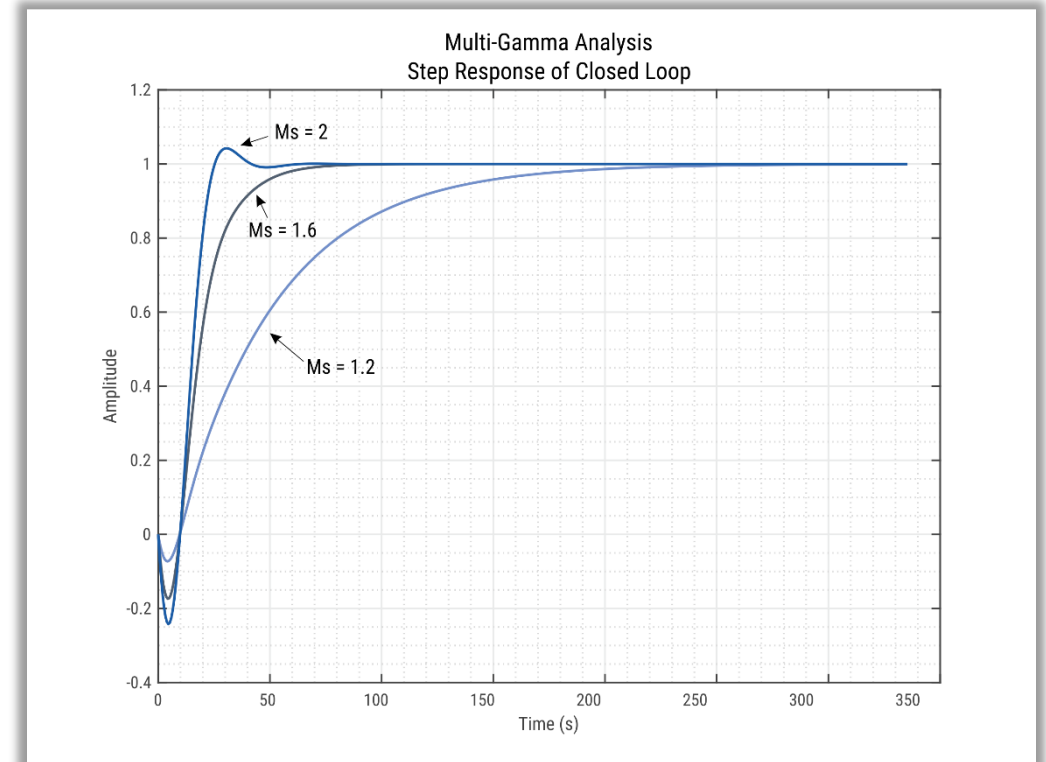
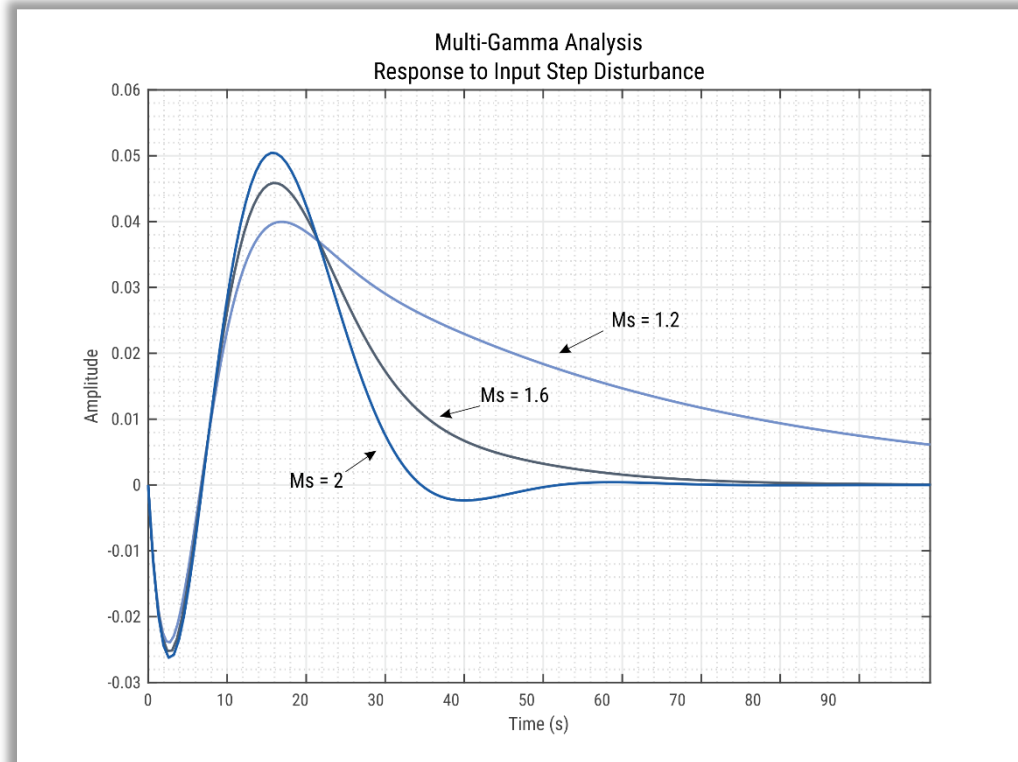


# Selection of Gamma ( $M_s$ ) Value



Amplitude frequency response of  
sensitivity function  $S(s)$  for IE optimal PI  
controllers

# Selection of Gamma ( $M_s$ ) Value



# Grid Connected Photovoltaic Inverter

**LCL filter with processing delay**  
for Single phase 3 kW Grid-  
Connected Photovoltaic Inverter  
system

$$P(s) = \frac{1}{1e - 4s + 1} \cdot \frac{s^2 + 1.143e4s + 1.587e8}{1.2e - 3s^3 + 21.71s^2 + 3.016e5s}$$

## Requirements

Controller:  $C_{PR}(s) = k_p + k_r \frac{2\omega_c s}{s^2 + 2\omega_c s + \omega_0^2}$

Design focus: Track a current 50Hz sinusoidal reference waveform (servo problem)

## Formulation of the design problem

Sensitivity functions:  $S(s) = \frac{1}{1 + C_{PR}(s)P(s)}$

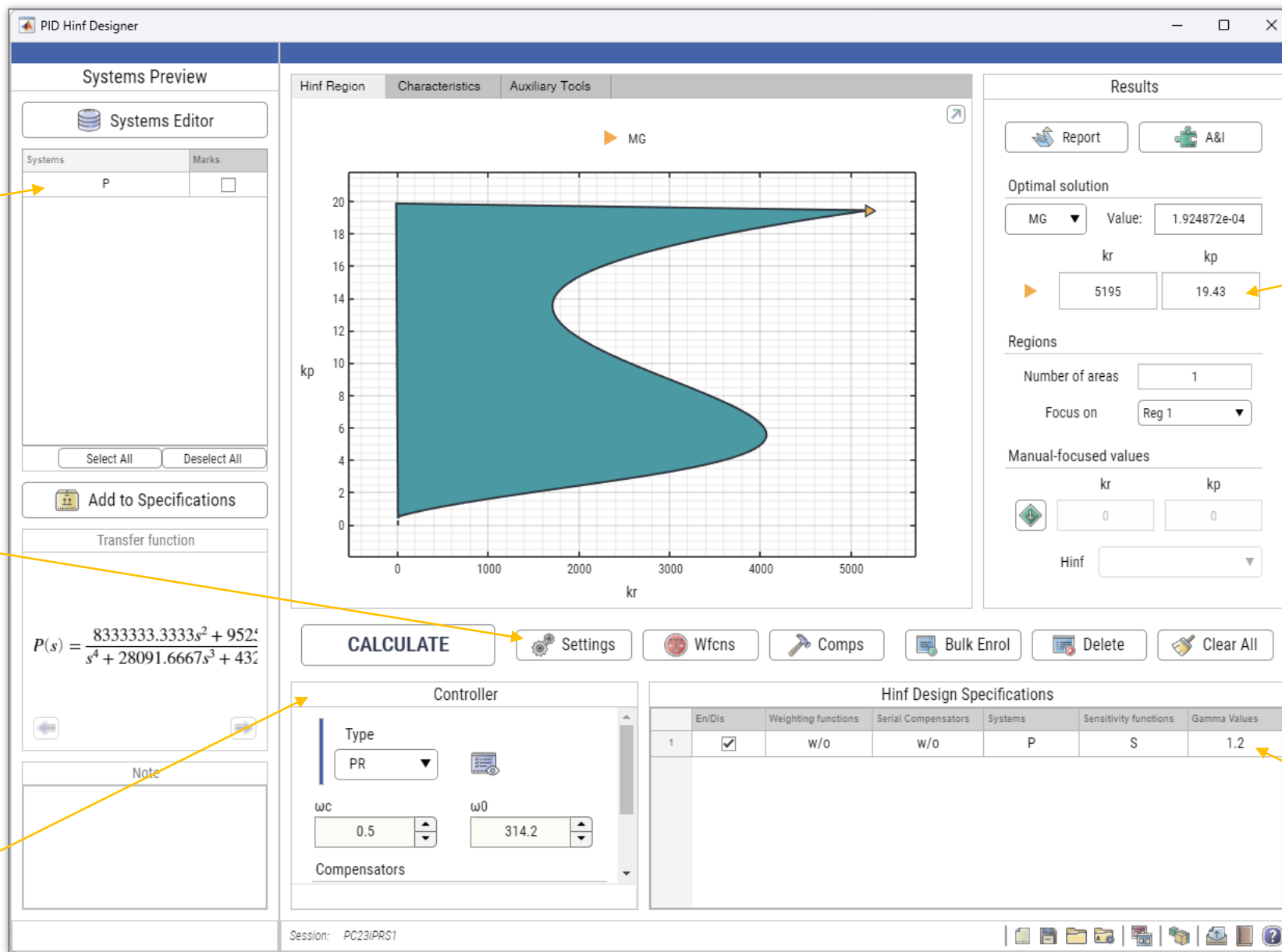
Weighting functions:  $W(s) = 1$

Design parameters:  $\omega_0 = 314.2 \text{ rad} / \text{s}, \omega_c = 0.5 \text{ rad} / \text{s}$

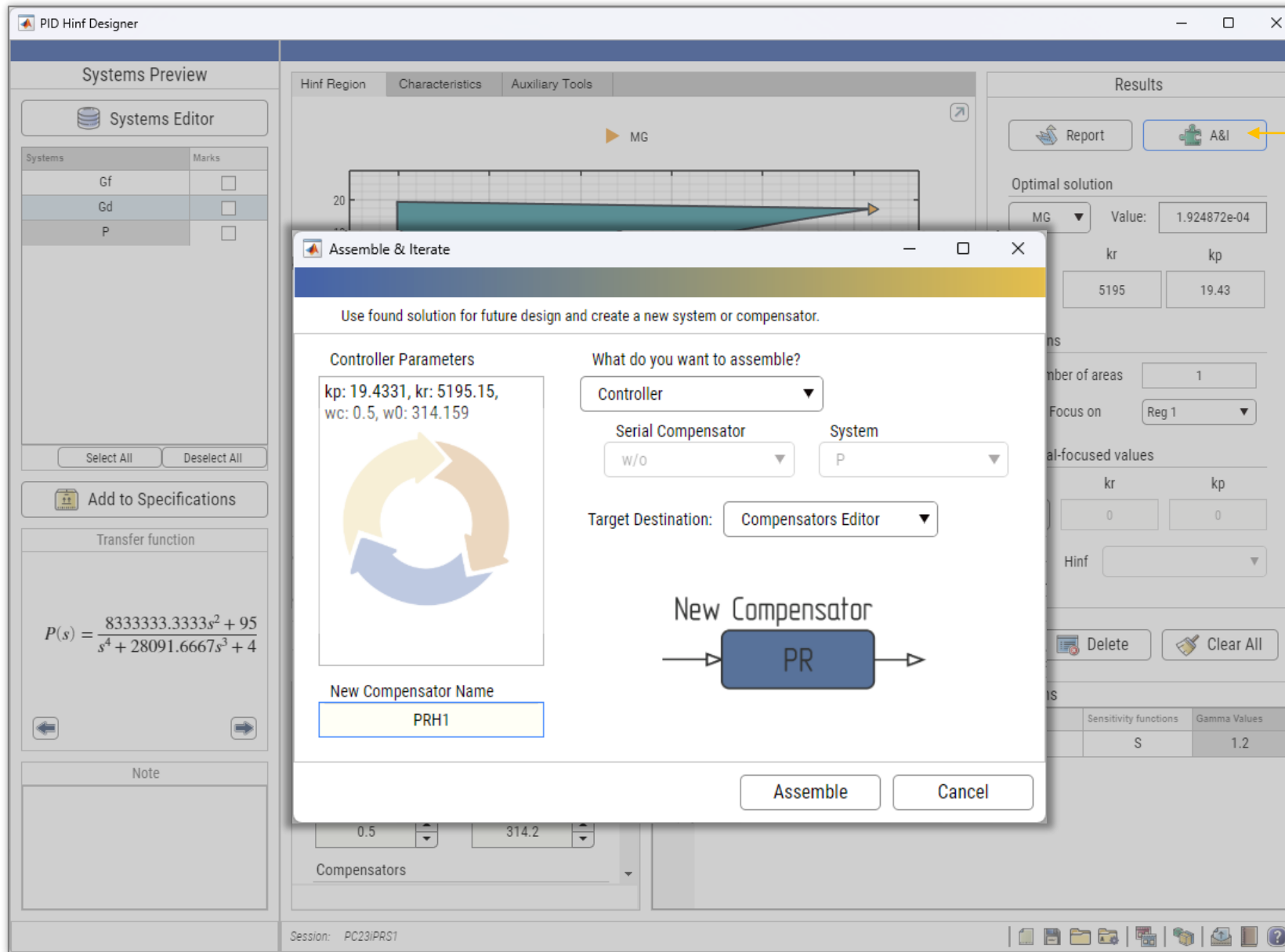
Design criterion :  $\max_{C_{PR}} k_r$



**PID**  
**H<sub>∞</sub> Designer**







Iterative modification  
of the controller

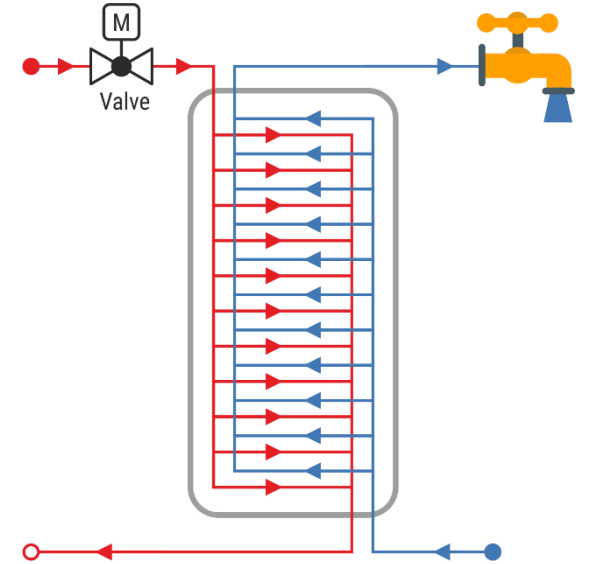
# Plate Heat Exchanger

Description of the process based on experimentally obtained Input/Output data

## Requirements

Controller:  $C_{PI}(s) = K \left( 1 + \frac{1}{T_{i,s}} \right) = k_p + \frac{k_i}{s}$

Design focus: Track reference temperature setpoint (servo problem)



## Formulation of the design problem

Sensitivity functions:  $S_i(s) = \frac{1}{1 + C_{PI}(s)P_i(s)}, \quad T_i(s) = \frac{C_{PI}(s)P_i(s)}{1 + C_{PI}(s)P_i(s)}, \quad i = 1, 2, 3$

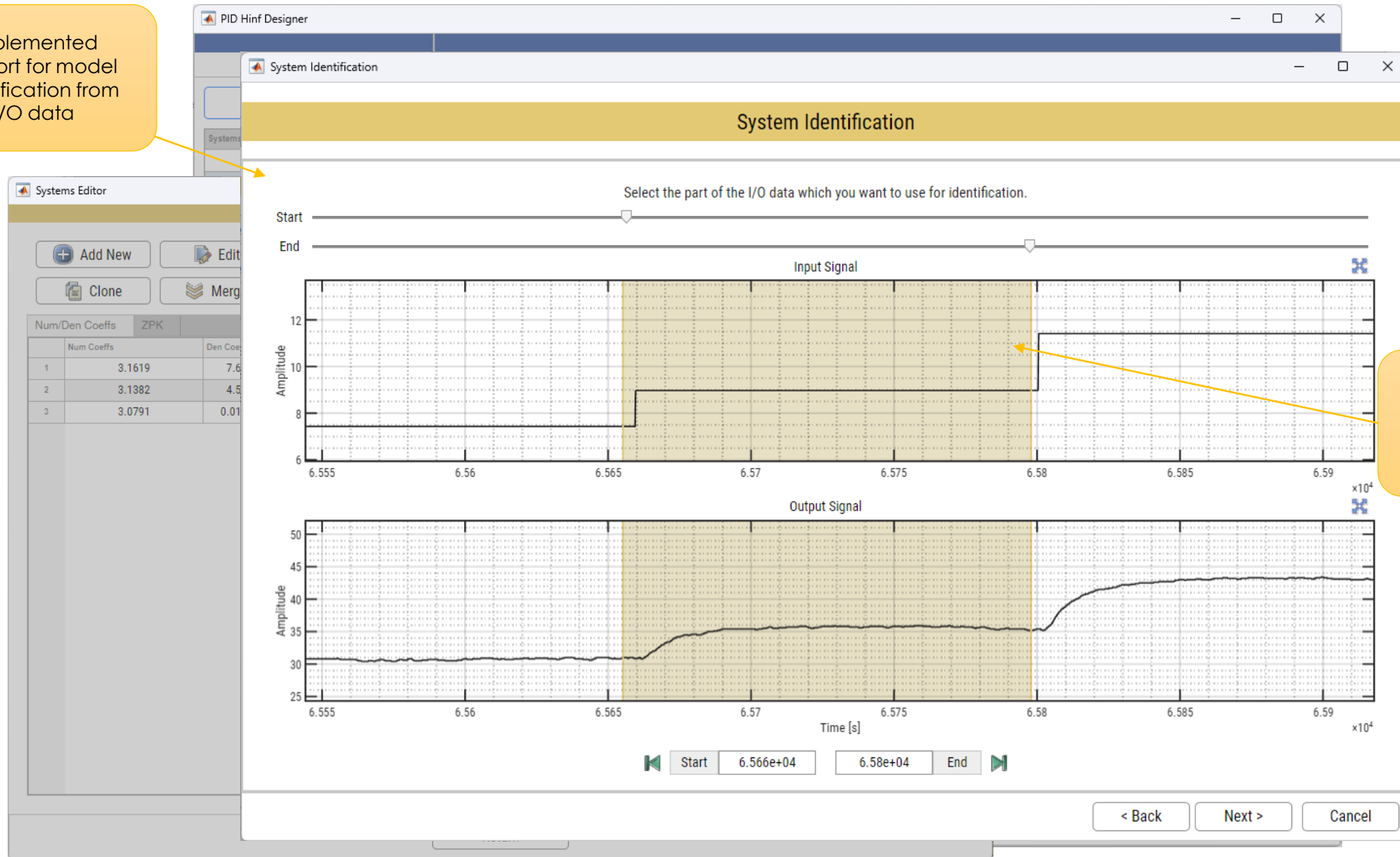
Weighting functions:  $W_{S_i}(s) = 1, \quad W_{T_i}(s) = 1, \quad i = 1, 2, 3$

Design criterion:  $\min_{C_{PI}} \max_{i \in \{1, 2, 3\}} \int_0^\infty |e_i(t)| dt, \quad \min_{C_{PI}} \max_{i \in \{1, 2, 3\}} \int_0^\infty t |e_i(t)| dt$

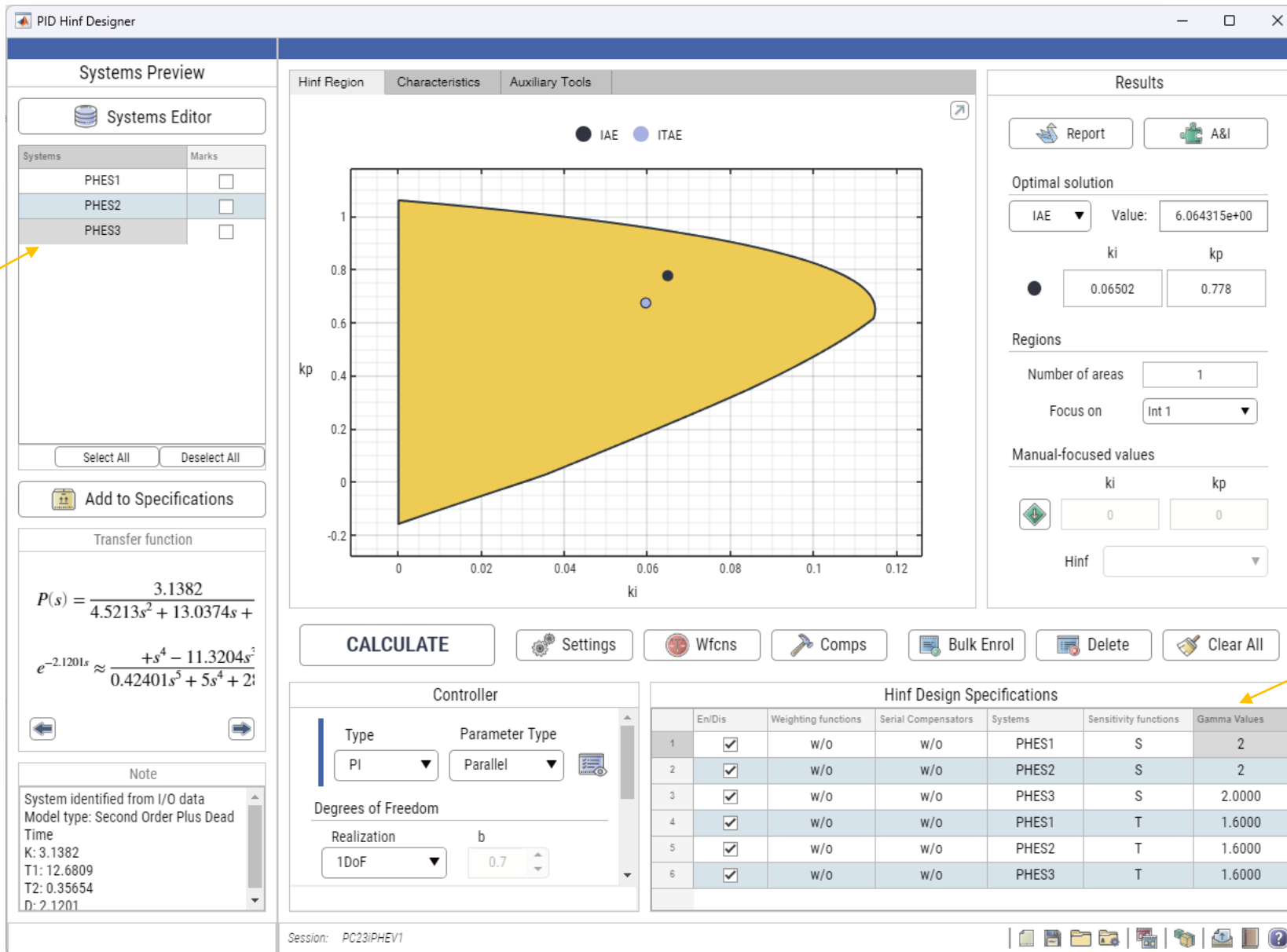


**PID**  
**H<sub>∞</sub> Designer**

Implemented support for model identification from I/O data



Identification of the model at different operating points of the process



Definition of identified models

$H_\infty$  limitations for S(s) and T(s) functions

# Conclusion

- New analytical method for the design of the  $H_\infty$  affine controller
- Affine controller includes almost all fixed structure controllers commonly used in practice

$$C(s, \mathbf{k}) \triangleq k_q Q(s) + k_r R(s) + F(s), \mathbf{k} \triangleq \begin{bmatrix} k_r, k_q \end{bmatrix}^T \in \mathbb{R}^2$$

- Web tool PID  $H_\infty$  Designer ([www.pidlab.com](http://www.pidlab.com))

