# Analytical Design of Controllers with Two Tunable Parameters Based on $H_\infty$ Specifications for Dead-Time Systems

Michal Brabec[1], Miloš Schlegel[2]

[1] *brabecm@ntis.zcu.cz - Department of Cybernetics, Faculty of Applied Sciences, University of West Bohemia, Pilsen, Czechia*
[2] *schlegel@ntis.zcu.cz - NTIS Research Center, Faculty of Applied Sciences, University of West Bohemia, Pilsen, Czechia*



**Abstract**
The main result of this paper is a new analytical method for the design of a wide class of controllers with two tunable and any number of fixed parameters based on $H_\infty$ specifications for SISO dead-time systems. The essence of the method lies in the analytical description of the boundary of the $H_\infty$ region in the parametric plane of the controller. In addition to the analytical method, the paper refers to a user-friendly web-based design tool available at *www.pidlab.com*, where this extended methodology is implemented for multiple process models and design requirements in the form of $H_\infty$ performance and robustness constraints. Two examples illustrate the practical applicability of the proposed approach.

**Keywords**
fixed order controller, $H_\infty$, parametric plane, PID $H_\infty$ Designer, dead-time systems

## Contents



## 1. Introduction

Dead time is a dominant aspect of many industrial systems. It can arise from various sources: system dynamics, feedback control, actuators, control interfaces, etc. This delay, inherent in chemical and biological processes, transportation systems, and communication networks, poses significant challenges to control systems due to its complex influence on stability and performance.

The characteristic function associated with the closed-loop system (see Fig. 1) can exhibit three types of systems: retarded, neutral, and advanced. In this paper, our focus is narrow to systems of the retarded type, particularly with a single fixed dead time. The infinite dimensionality of the delay element $e^{-hs}$ is a primary source of the numerous technical challenges associated with dead-time systems.

One of the simple ways to solve problems with the infinite-dimensional delay element $e^{-hs}$ is its approximation by finite-dimensional component, which enables the usage of standard analysis techniques and design methods. Widely adopted techniques like Pade approximation are mostly effective, particularly in the low-frequency range for fixed dead time cases.

When considering different approaches for the design of controllers, it is essential to mention the usage of the $\mathcal{H}_\infty$ norm to specify design requirements. The design of optimal $\mathcal{H}_\infty$ controllers for LTI systems is traditionally based on the Riccati equation or linear matrix inequality (LMI) ([10]). This results in generating controllers with complex high-order dynamics compared to the plant. Moreover, these solutions are pretty fragile ([18]). The traditional $\mathcal{H}_\infty$ synthesis lacks mechanisms to impose a structure of controllers. Structure $\mathcal{H}_\infty$ synthesis is an alternative to this approach. But the design of arbitrary fixed-structure controllers leads to a complex, nonconvex, and nonsmooth optimisation NP-hard problem, potentially involving discontinuities within the solution space ([1]). If we additionally consider that the system includes dead time, the problem further deepens.

Different publications use various optimisation-based techniques for such design problems. In [16], a method based on the generalised Hermite-Biehler theorem is presented. It determines all stabilising PID controllers that ensure the internal stability of the closed-loop system. Subsequently, a linear programming-based algorithm is utilised to determine admissible $H_\infty$ PID controllers that meet the specified criteria. However, it can be said that, in general, optimisation-based techniques for dead-time systems aim to identify controllers that stabilise the system while minimising the $H_\infty$ norm of the associated transfer function. The priority is then to determine the stabilising controller first. Further optimisation can be undertaken while meeting the stability condition. The different iconic approaches interested in the stabilisation problem for dead-time systems are discussed in [17, 19, 24].

If we look at the infinite-dimensional $H_\infty$ optimisation problem, in [15], the fixed-order $H_\infty$ controllers are designed for a specific class of time-delay systems. A nonsmooth, nonconvex optimisation approach is employed, and a method for computing the $H_\infty$ norm ([14]). The optimisation process contains two main steps: stabilisation to minimise spectral abscissa and $H_\infty$ optimisation using a hybrid optimisation method ([13]). An alternative path was taken in ([2]), wherein a nonsmooth trust-region bundle method is utilised to compute locally optimal $H_\infty$ controllers for a frequency-sampled approximation of the underlying infinite-dimensional $H_\infty$ problem. Optimisation then relies on the nonsmooth trust-region method ([3, 4]), with system transfer functions discretised across a fine grid of frequencies.

Such methods typically produce only one locally optimal solution, but it's necessary to say that not every time is required to minimise the $\mathcal{H}_\infty$ norm magnitude. Providing all solutions that satisfy given constraints is also a crucial feature from a practical point of view. If the interest is shifted to the case where the desired controller has only two or three adjustable parameters, the parametric space approach can be employed. The solution to the design problem can then be represented by a region in the parametric plane of the controller. Points of this region represent all controllers that meet specified design requirements. In [12], the foundations for such an approach were outlined through the D-decomposition paradigm defining the admissible solutions region.

This paper describes the analytical design method of the $H_\infty$ affine controllers using $H_\infty$ specifications in the presence of a system with dead time. Such a controller structure enables the characterisation of a wide class of controllers with two tunable and any number of fixed parameters. Building on previous work ([8, 22]), our approach provides an analytical, computational procedure for translating diverse $H_\infty$ design specifications to the boundary of the $H_\infty$ region in the parametric plane of the controller. Through the intersection of $H_\infty$ regions, it is possible to implement both single and multiple system model designs and, at the same time, demand fulfilment of several $H_\infty$ limitations. In contrast to the previously published technique in [8] that was dependent on Pade approximation for dead-time systems, we introduce an upgraded analytical, computational procedure that doesn't rely on such simplification. Due to the frequency domain-based design approach, the method met no significant hurdles. However, a problem arose during a step involving standard stability analysis techniques due to the infinite dimensionality of the delay element $e^{-hs}$. This fact led to the revision of the original design procedure ([8]) and the proper selection of a method for stability analysis of systems with dead time.

The remainder of this paper is organised as follows: Section 2 presents the basic assumptions and a brief theoretical basis of the research problem. Section 3 is dedicated to formulating the analytical design method for dead-time systems. Section 4 contains the overview of the PID $H_\infty$ Designer and two exemplary examples. The paper's conclusion is in Section 5.

## 2. Preliminaries and Problem Statements

### 2.1 Control Loop

Consider a closed-loop control system shown in Fig. 1, where

$$P(s) = P_r(s)e^{-hs} \tag{1}$$

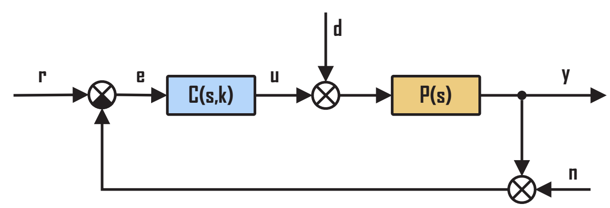


**Figure 1.** A standard feedback system

is a transfer function describing an LTI SISO plant, $P_r(s)$ is a strictly proper rational transfer function, and $h > 0$. The fixed structure controller $C(s, \mathbf{k})$ is assumed in the form of an affine function

$$C(s, \mathbf{k}) \triangleq k_q Q(s) + k_r R(s) + F(s), \tag{2}$$

where $Q(s)$, $R(s)$, and $F(s)$ are arbitrary proper rational transfer functions. The controller's transfer function (2) is affinely dependent on the two tunable parameters $k_q$ and $k_r$, hereafter referred to as the vector parameter $\mathbf{k} \triangleq [k_r, k_q]^T \in \mathbb{R}^2$. Note that under the given assumptions, the open-loop transfer function is in the form

$$L(s, \mathbf{k}) = C(s, \mathbf{k})P(s) \triangleq L_r(s, \mathbf{k})e^{-hs}, \tag{3}$$

where $L_r(s, \mathbf{k}) = \frac{L_{rn}(s,\mathbf{k})}{L_{rd}(s,\mathbf{k})}$ is the strictly proper rational transfer function, and consequently, the closed-loop system can be classified as retarded system. The choice of the controller transfer functions $Q(s)$, $R(s)$, and $F(s)$ is part of the design problem. Let's note that we can obtain practically all types of common controllers used in industry with a suitable selection of transfer functions $Q(s)$, $R(s)$, and $F(s)$ (for more details, see [8]). E.g. selection

$$Q(s) = \frac{s}{\tau s + 1}, \quad R(s) = \frac{1}{s}, \quad F(s) = k_p, \tag{4}$$

corresponds to a PID controller with a fixed proportional gain $k_p$ in the form

$$C_{PID}(s) = k_p + k_r \frac{1}{s} + k_q \frac{s}{\tau s + 1}. \tag{5}$$

The closed-loop robustness and control performance requirements can advantageously be formulated as constraints on the $H_\infty$ norm of one or more weighted sensitivity functions. For this purpose, let us consider the following sensitivity functions:

$$T(s, \mathbf{k}) = \frac{C(s, \mathbf{k})P(s)}{1 + C(s, \mathbf{k})P(s)}, \tag{6}$$

$$S(s, \mathbf{k}) = 1 - T(s, \mathbf{k}), \tag{7}$$

$$S_p(s, \mathbf{k}) = P(s)S(s, \mathbf{k}), \tag{8}$$

$$S_c(s, \mathbf{k}) = C(s, \mathbf{k})S(s, \mathbf{k}). \tag{9}$$

### 2.2 Iterative improvement of control performance by increasing controller complexity

Consider the case where it is requested to enhance the existing controller $C_0(s)$. In such a case, $F(s) = C_0(s)$ can be set. The improved controller is considered in the form (2), where $Q(s)$ and $R(s)$ can be chosen appropriately to improve the closed-loop performance. For example, in the first step, we will design a PD controller, and in the second step, we will supplement it with a PI controller connected in parallel. This way, we can design a PID controller as an alternative to the above mentioned method.

### 2.3 $H_\infty$ design specification

As shown in [11], various performance and robustness specifications could be made using the $H_\infty$ norm of weighted versions of the transfer functions (6–9). Consider the weighted sensitivity function

$$H(s, \mathbf{k}) = W(s)S_\star(s, \mathbf{k}), \tag{10}$$

where $S_\star(s, \mathbf{k})$ is an arbitrary closed-loop sensitivity function (6–9), and $W(s)$ is a stable weighting function. Closed-loop design requirements can now be expressed in a unified form as a restriction on the $H_\infty$ norm of the transfer function

$$|H(j\omega, \mathbf{k})| \le \gamma, \quad \forall \omega \in [0, \infty], \tag{11}$$

or equivalently

$$\|H(s, \mathbf{k})\|_\infty \le \gamma, \tag{12}$$

where $\|H\|_\infty \triangleq \sup_\omega |H(j\omega)|$ is called $H_\infty$ norm.

## 3. Basic Design Problem for Dead-Time Systems

The fundamental objective in the design of the $H_\infty$ affine controller (2) is to identify all stabilising controllers that ensure the internal stability of the closed-loop system and meet the performance/robustness criterion (12). If the dead time value $h$ is zero or the infinite-dimensional delay element $e^{-hs}$ is approximated by a finite-dimensional LTI system, standard stability analysis techniques can be applied. In the case of an actual dead time, it is necessary to properly select the effective method for automatic stability evaluation of systems with dead time. For the purposes of this article, two different methods, described in section (3.5), are discussed.

Consider that $H(s, \mathbf{k})$, from (10) is in the form

$$H(s, \mathbf{k}) = \frac{H_n(s, \mathbf{k})}{H_d(s, \mathbf{k})}, \tag{13}$$

where $H_n(s, \mathbf{k})$ and $H_d(s, \mathbf{k})$ are quasi-polynomials with real coefficients. Similarly, we can rewrite the controller (2) to

$$C(s, \mathbf{k}) = k_q \frac{Q_n(s)}{Q_d(s)} + k_r \frac{R_n(s)}{R_d(s)} + \frac{F_n(s)}{F_d(s)}. \tag{14}$$

Let $\mathcal{K}$ is the set of all controller parameters $\mathbf{k} = [k_r, k_q]$ satisfying the performance/robustness criterion (12).

$$\mathcal{K} = \Big\{ \mathbf{k} \in \mathbb{R}^2 : H(s, \mathbf{k}) \textit{ is asymptotically stable} \wedge \Big| H_n(j\omega, \mathbf{k}) \Big| \le \gamma \cdot \Big| H_d(j\omega, \mathbf{k}) \Big|, \omega \in [0, \infty) \Big\}. \tag{15}$$

We will hereafter call this set the $H_\infty$ *region* and the controller $C(s, \mathbf{k})$ for $\mathbf{k} \in \mathcal{K}$ the $H_\infty$ *controller*. We first find a set of certain curves that contain the boundary $\delta\mathcal{K}$ of the $H_\infty$ region $\mathcal{K}$ as its subset. For this purpose, the Theorem from [12] will be used. Under the above assumptions, it has the following form.

**Theorem 1.** *The boundary of the set $\mathcal{K}$ is contained in the solution of the systems*

$$\begin{cases} H_n(j\omega, \mathbf{k}) = 0, & \text{(16a)} \\ H_d(j\omega, \mathbf{k}) = 0, & \text{(16b)} \end{cases}$$

$$\begin{cases} |H(j\omega,\mathbf{k})|^2 = \gamma^2, & (17a)\\ \dfrac{\partial |H(j\omega,\mathbf{k})|^2}{\partial \omega} = 0, & (17b)\end{cases}$$

*for $\omega \in \langle 0, +\infty)$ and three equations*

$$|H(0,\mathbf{k})| = \gamma, \tag{18}$$

$$|H(j\infty,\mathbf{k})| = \gamma, \tag{19}$$

An analysis of the *Theorem 1* systems allows the derivation of an analytical method for determining the boundary of $H_\infty$ region. The following statements are given only for the case of the weighted sensitivity function $S(s)$ (7). However, they can be extended for other closed-loop functions, using the resultant theorem ([7]). Their proofs follow the analogous rationale as for systems without dead time and are explained in detail in [8]. Therefore, they are omitted.

Let us consider the the strictly proper rational transfer function $P_r(s)$ and the weighting function $W(s)$ in the rational coprime form $P_r(s) = \frac{P_{rn}(s)}{P_{rd}(s)}$ and $W(s) = \frac{W_n(s)}{W_d(s)}$, respectively. The controller structure is intended in the affine form (14). For the case of $H(s,\mathbf{k}) = W(s)S(s,\mathbf{k})$ then it holds that

$$H(s,\mathbf{k}) = \frac{W_n(s)Q_d(s)R_d(s)F_d(s)P_{rd}(s)}{H_d(s,\mathbf{k})}, \tag{20}$$

where

$$\begin{aligned} H_d(s,\mathbf{k}) = & W_d(s)\big(F_d(s)P_{rn}(s)e^{-hs}Q_d(s)R_n(s)k_r + \\ & F_d(s)P_{rn}(s)e^{-hs}Q_n(s)R_d(s)k_q + \\ & Q_d(s)R_d(s)F_d(s)P_{rd}(s) + \\ & F_n(s)P_{rn}(s)e^{-hs}Q_d(s)R_d(s)\big). \end{aligned} \tag{21}$$

### 3.1 Analysis of the system (16)

**Lemma 1.** *$H_n(j\tilde{\omega},\boldsymbol{k}) = 0$, $\tilde{\omega} \in \mathbb{R}$, if and only if one of the following two conditions holds:*

*(i) At least one of the transfer functions $Q(s)$, $R(s)$, $F(s)$, and $P(s)$ has a pole $j\tilde{\omega}$, $\tilde{\omega} \in \mathbb{R}$ on the imaginary axis of the complex plane.*

*(ii) The weighting function $W(s)$ has zero $j\tilde{\omega}$, $\tilde{\omega} \in \mathbb{R}$ on the imaginary axis of the complex plane.*

**Lemma 2.** *Let $\tilde{\omega} \in \mathbb{R}$ satisfies Lemma 1 and assume that $W_d(j\tilde{\omega}) \neq 0$ then the system (16) has a solution iff there exists $\tilde{\boldsymbol{k}} = [\tilde{k}_r, \tilde{k}_q]$ such that $H_d(j\tilde{\omega},\tilde{\boldsymbol{k}}) = 0$ or equivalently iff at least one of the following conditions is true*

*(i)* $\tilde{k}_q \cdot F_d(j\tilde{\omega})P_{rn}(j\tilde{\omega})e^{-hj\tilde{\omega}}Q_n(j\tilde{\omega})R_d(j\tilde{\omega}) = 0,$

*(ii)* $\tilde{k}_r \cdot F_d(j\tilde{\omega})P_{rn}(j\tilde{\omega})e^{-hj\tilde{\omega}}Q_d(j\tilde{\omega})R_n(j\tilde{\omega}) = 0,$

*(iii)* $F_n(j\tilde{\omega})P_{rn}(j\tilde{\omega})e^{-hj\tilde{\omega}}Q_d(j\tilde{\omega})R_d(j\tilde{\omega}) = 0,$

*(iv)*
$$\begin{aligned} & \tilde{k}_q \cdot F_d(j\tilde{\omega})P_{rn}(j\tilde{\omega})e^{-hj\tilde{\omega}}Q_n(j\tilde{\omega})R_d(j\tilde{\omega}) + \\ + \ & \tilde{k}_r \cdot F_d(j\tilde{\omega})P_{rn}(j\tilde{\omega})e^{-hj\tilde{\omega}}Q_d(j\tilde{\omega})R_n(j\tilde{\omega}) + \\ + \ & F_n(j\tilde{\omega})P_{rn}(j\tilde{\omega})e^{-hj\tilde{\omega}}Q_d(j\tilde{\omega})R_d(j\tilde{\omega}) = 0, \end{aligned}$$

*(v)* $H_d(j\tilde{\omega},\tilde{\boldsymbol{k}})/W_d(j\tilde{\omega}) = 0.$

### 3.2 Analysis of the system (17)

**Lemma 3.** *Assume that (16b) does not hold, then the equation (17a) can be expressed equivalently in the form*

$$p_1(\omega,\boldsymbol{k}) = 0, \tag{22}$$

*where $p_1(\omega,\boldsymbol{k})$ is a second-order polynomial with real coefficients in the variables $k_r$ and $k_q$.*

**Lemma 4.** *Assume that (16b) does not hold, then the equation (17b) can be expressed equivalently in the form*

$$p_2(\omega,\boldsymbol{k}) = 0, \tag{23}$$

*where $p_2(\omega,\boldsymbol{k})$ is a second-order polynomial with real coefficients in the variables $k_r$ a $k_q$.*

The solution of the system (17) can be determined analytically by converting (22) and (23) to an algebraic equation of the fourth degree with one unknown (for more details, see [8]).

### 3.3 Analysis of the equations (18) and (19)

The equation (18) and (19) are equivalent to the equations

$$p_1(0,\mathbf{k}) = 0 \quad \text{and} \quad \lim_{\omega\to\infty} p_1(\omega,\mathbf{k}) = 0,$$

where $p_1(\omega,\mathbf{k})$ is the second-order polynomial from *Lemma 3*.

### 3.4 Sketch of $H_\infty$ region isolation algorithm

*Step 1:* Based on the *Theorem 1*, we can identify all points in the parametric plane of the controller $C(s,\mathbf{k})$ that are suspected to form the boundary $\delta\mathcal{K}$. These points lie on a finite number of curves outlined by equations (16-19) in *Theorem 1*. The ensemble of these curves is denoted as $\mathcal{B}$.

*Step 2:* From the curves of the set $\mathcal{B}$, established in step 1, we isolate those segments that fulfil both the $H_\infty$ specification (12) and the criterion for internal stability. Such selected segments are represented by $\mathcal{B}^*$.

*Step 3:* The curve segments of the set $\mathcal{B}^*$, from step 2, form the boundary of one or multiple regions within the controller's parametric plane. These regions can be bounded or unbounded. The union of these regions determines the searched set $\mathcal{K}$, including all $H_\infty$ controllers satisfying the elementary $H_\infty$ specification (12).

### 3.5 Stability Analysis of Dead-Time Systems

The points of segments from (3.4-Step 2), representing the set of unique parameters of the proposed controller, must be subjected to an internal stability analysis during the design procedure. Here, we offer a concise overview of two approaches for evaluating the stability of systems with dead time. The primary method employed in our approach offers practical, intuitive, and numerically efficient solutions for stability analysis of retarded system.

*Delay sweeping* ([21])

Consider the characteristic quasi-polynomial $\mathcal{X}_h(s, \mathbf{k})$ of the closed-loop system from Fig. 1, expressed as

$$\mathcal{X}_h(s, \mathbf{k}) = L_{rd}(s, \mathbf{k}) + L_{rn}(s, \mathbf{k})e^{-hs}, \tag{24}$$

where $L_{rd}(s)$ and $L_{rn}(s)$ denote the denominator and numerator, respectively, of the strictly rational part $L_r(s, \mathbf{k})$ of the open-loop transfer function $L(s, \mathbf{k})$. This quasi-polynomial has an infinite number of roots for $h > 0$. To ensure closed-loop stability, additional necessary conditions must be met: $L_{rn}(s, \mathbf{k})$ and $L_{rd}(s, \mathbf{k})$ have no common roots in closed right half-plain, and $L_{rn}(0, \mathbf{k}) + L_{rd}(0, \mathbf{k}) \neq 0$ ([21]). The utilised stability analysis technique is based on the direct Walton-Marshall method ([20]), which uses the fundamental property of (24) where its roots continuously change with $h$ for $h \geq 0$ ([9]). As $h$ varies in $\mathbb{R}^+$, root migrations occur just between the left-half plane and the right-half plane through the imaginary axis. The analysis begins by employing conventional methods to determine the unstable roots of (24) at $h = 0$. Subsequently, as $h$ increases, the imaginary axis crossings are counted, with each transition from left to right incrementing the count of unstable poles, and vice versa ([21]).

The second method stands out as an alternative way that represents a sophisticated toolkit with a broader range of capabilities compared to the previous method.

*TDS-CONTROL package* ([6])

It is a comprehensive MATLAB package for the analysis and controller design of LTI time-delay systems characterised in state-space form ([5]). It can handle both retarded and neutral time-delay systems, as well as specific systems described in delay descriptor form. The package enables various analysis methods, including the computation of spectral abscissa, $H_\infty$ norm, pseudospectral abscissa, and distance to instability. For our purposes, the calculation of spectral abscissa is essential. To compute the roots of characteristic functions in the right half-plane, the package implements a specialised algorithm ([23]) combining a spectral discretisation of the infinitesimal generator of the solution operator associated with the closed-loop system with Newton correction based on the nonlinear characteristic function ([6]).

## 4. Application Examples

In this section, the software tool PID $H_\infty$ Designer is briefly introduced, and two illustrative examples declaring the versatility of the analytical design method are provided here.

### 4.1 Overview of PID $H_\infty$ Designer

PID $H_\infty$ Designer is a sophisticated MATLAB-based software developed to analyse, design, and tune two-parameter-constrained affine controllers. Still, it is adapted for both the general and particular controller forms (like PI, PID, Proportional-Resonant, etc.). Likewise, this versatile tool offers solutions for both simple and complex design problems like cascade and repetitive control. The design flexibility is enhanced by extending the proposed controller with fixed series or parallel compensators. Moreover, it supports model sets derived from commonly used identification experiments, including the non-standard moment model set provided by the PIDMA-autotuner from REX Controls. The tool offers two distinct design environments: “Step By Step” and “Workspace”, which offer a dynamic and interactive environment suitable for novice and expert users. The “Step By Step” environment guides beginners through the step-by-step design procedure, while the “Workspace” environment offers plenty of advanced tools for more experienced users. The software demo version and further resources with examples are accessible at *www.pidlab.com*.

### 4.2 Example 1

Consider unstable FOPDT plants of the form

$$P(s) = \frac{1}{s-1}e^{-0.3s}. \tag{25}$$

For this system, we want to find all one degree of freedom (1DoF) stabilising PI controllers,

$$C_{PI}(s) = k_p + \frac{k_i}{s}, \tag{26}$$

meeting the two following design requirements on (7) and (6) according to (12):

$$\|S(s)\|_\infty \leq \gamma, \tag{27}$$

$$\|T(s)\|_\infty \leq \gamma, \tag{28}$$

where $\gamma = 2.5$. We want to select the one with the maximum gain of the integration part from these controllers.

Applying the design algorithm described above, we obtain two $H_\infty$ regions from constraints (27) and (28). Their intersection is the desired $H_\infty$ region (see Fig. 2). The resulting 1DoF controller with the maximum gain of the integration component has the parameters $k_p = 2.03, k_i = 0.3575$. The corresponding Nyquist curve of the open-loop transfer function is depicted in Fig. 3. The sensitivity and complementary sensitivity functions are shown in Fig. 4, and the closed-loop responses for the two degrees of freedom (2DoF) controller ([25]), with the weighting factor $b = 0.35$, to the step in the set-point and the input disturbance are shown in Fig. 5. Note that for the design parameter $\gamma < 2.1$, the given design problem has no solution.

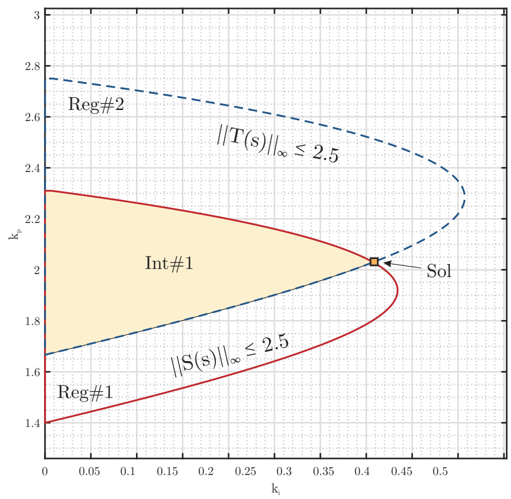


**Figure 2.** Example 1: $H_\infty$ region (—Reg#1) of (27) and $H_\infty$ region (- -Reg#2) of (28) with their intersection ( Int#1) and optimal solution ( Sol).

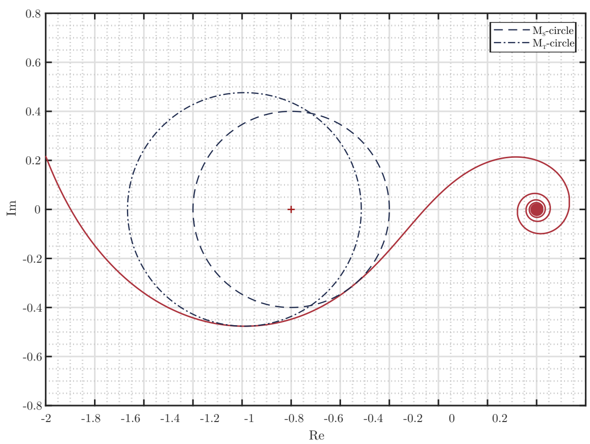


**Figure 3.** Example 1: The Nyquist curve of the open-loop transfer function (3) with $M_S$ and $M_T$ circles ([25]).

### 4.3 Example 2

Consider a simple inverted mathematical pendulum on a cart, where the input is considered the cart's acceleration and, as the output, the angle of deflection of the pendulum. It can be shown that near the unstable equilibrium point, we can describe this system by a transfer function

$$P(s) = \frac{-ml}{ml^2 s^2 + bs - gml}. \tag{29}$$

To design a stabilising PID controller, let's consider a transfer function in the form ($m = 0.1, g = 9.81, b = 0.0, l = 0.50$)

$$P(s) = \frac{0.05}{-0.025s^2 + 0.4905} e^{hs}, \tag{30}$$

where $h = 0.01$ represents the dead time in the feedback control loop. We want to find all stabilising PID controllers

$$C_{PID}(s) = k_p + \frac{k_i}{s} + \frac{k_d s}{\tau s + 1}, \tag{31}$$

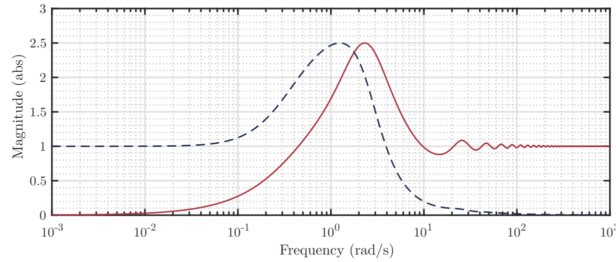


**Figure 4.** Example 1: The amplitude-frequency response of sensitivity function $S(s)$ (—) and complementary sensitivity function $T(s)$ (- -).

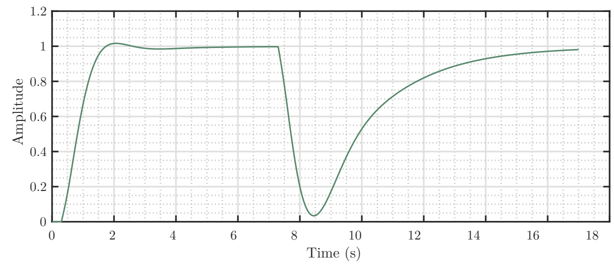


**Figure 5.** Example 1: The closed-loop response to step in the setpoint and the input disturbance.

where $k_p$ and $k_i$ are tunable parameters, and $k_d = -10$, $h = 0.005$, meeting the design requirements (27) and (28) for $\gamma = 1.33$. The optimal solution with a maximum gain of the integration component then occurs for the following parameters $k_p = -48.57, k_i = -30.58$.

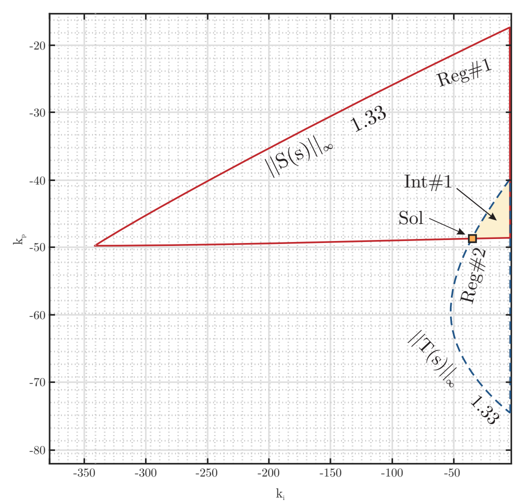


**Figure 6.** Example 2: $H_\infty$ region (—Reg#1) of (27) and $H_\infty$ region (- -Reg#2) of (28) with their intersection ( Int#1) and optimal solution ( Sol).

## 5. Conclusion

This paper introduces an upgraded analytical approach for the design of the $H_\infty$ affine controllers based on $H_\infty$ specifications for dead-time systems. Such a controller structure enables the characterisation of a wide class of controllers with two adjustable and any number of fixed parameters. Unlike previously published method that depends on Pade approximation for dead-time systems, the presented approach here shows an enhanced analytical, computational procedure that eliminates the need for such simplifications. By employing this upgraded method, we offer a robust technique for designing controllers that can effectively handle the challenges of dead time in control systems.

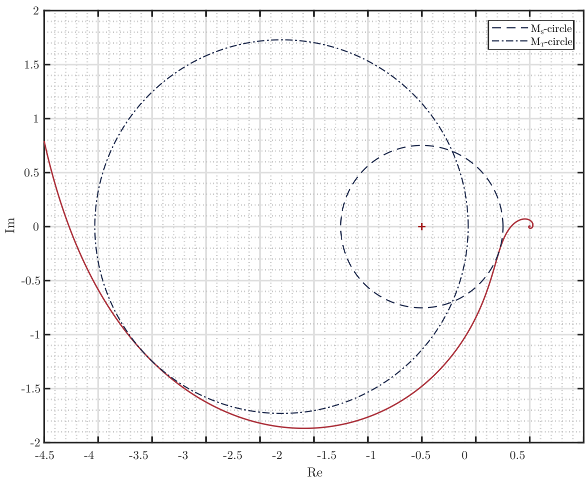


**Figure 7.** Example 2: The Nyquist curve of the open-loop transfer function (3) with $M_S$ and $M_T$ circles ([25]).

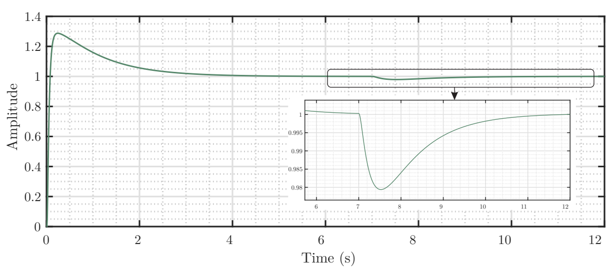


**Figure 8.** Example 2: The closed-loop response to step in the setpoint and the input disturbance.

## References


[1] P. Apkarian and D. Noll. The $\mathcal{H}_\infty$ control problem is solved. *AerospaceLab Journal, ONERA*, 2017.

[2] P. Apkarian and D. Noll. Structured h-infinity control of infinite dimensional systems. *International Journal of Robust and Nonlinear Control*, 2017.

[3] P. Apkarian, D. Noll, and L. Ravanbod. Computing the structured distance to instability. 2015.

[4] P. Apkarian, D. Noll, and L. Ravanbod. Nonsmooth bundle trust-region algorithm with applications to robust stability. *Set-Valued and Variational Analysis*, 2016.

[5] P. Appeltans and W. Michiels. *Analysis and controller-design of time-delay systems using TDS-CONTROL. A tutorial and manual.* 2023.

[6] P. Appeltans, H. Silm, and W. Michiels. Tds-control: a matlab package for the analysis and controller-design of time-delay systems. *IFAC-PapersOnLine*, 2022.

[7] D. S. Bernstein. *Matrix Mathematics : Theory, Facts, and Formulas with Application to Linear Systems Theory*. N.J.: Princeton University Press, 2005.

[8] M. Brabec and M. Schlegel. Analytical design of a wide class of controllers with two tunable parameters based on $\mathcal{H}_\infty$ specifications. *2023 24th International Conference on Process Control (PC)*, 2023.

[9] R. Datko. A procedure for determination of the exponential stability of certain differential-difference equations. *Quart. Appl. Math.*, 1978.

[10] J. Doyle, K. Glover, P. Khargonekar, and B. Francis. State space solution to standard $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control problem. *Automatic Control, IEEE Transactions on*, pages 831 – 847, 1989.

[11] J. C. Doyle, B. A. Francis, and A. R. Tannenbaum. *Feedback Control Theory*. New York: Macmillan Publishing Company, 1992.

[12] E. Gryazina, B. Polyak, and A. Tremba. D-decomposition technique state-of-the-art. *Automation and Remote Control*, pages 1991–2026, 2008.

[13] S. Gumussoy, D. Henrion, M. Millstone, and M. L. Overton. Multiobjective robust control with hifoo 2.0. *IFAC Proceedings Volumes*, 42(6):144–149, 2009. 6th IFAC Symposium on Robust Control Design.

[14] S. Gumussoy and W. Michiels. Computing h-infinity norms of time-delay systems. *Proceedings of the IEEE Conference on Decision and Control*, 2010.

[15] S. Gumussoy and W. Michiels. Fixed-order h-infinity optimization of time-delay systems. *Recent Advances in Optimization and its Applications in Engineering*, pages 103–112, 2010.

[16] M. T. Ho. Synthesis of $\mathcal{H}_\infty$ pid controllers: A parametric approach. *Automatica*, 39(6):1069–1075, 2003.

[17] N. Hohenbichler. All stabilizing pid controllers for time delay systems. *Automatica*, 2009.

[18] L. H. Keel and S. P. Bhattacharyya. Robust, fragile, or optimal? *IEEE Transactions on Automatic Control*, 42(8):1098–1105, 1997.

[19] X. G. Li, S. I. Niculescu, J. X. Chen, and T. Chai. Characterizing pid controllers for linear time-delay systems: A parameter-space approach. *IEEE Transactions on Automatic Control*, 2021.

[20] J. E. Marshall, H. Gorecki, A. Korytowski, and K. Walton. *Time-delay systems: stability and performance criteria with applications*. New York (N.Y.): Ellis Horwood, 1992.

[21] L. Mirkin. *Time Delays in Control Systems - Course notes*. Faculty of Mechanical Engineering, Technion, 2021.

[22] M. Schlegel and P. Medvecová. Design of pi controllers: $\mathcal{H}_\infty$ region approach. *IFAC-PapersOnLine*, pages 13–17, 2018.

[23] Z. Wu and W. Michiels. Reliably computing all characteristic roots of delay differential equations in a given right half plane using a spectral method. *Journal of Computational and Applied Mathematics*, 2012.

[24] H. Xu, A. Datta, and S.P. Bhattacharyya. Pid stabilization of lti plants with time-delay. pages 4038–4043 vol.4, 2003.

[25] K. J. Åström and T. Hägglund. *Advanced PID control*. ISA, Advanced PID control, 2006.